# 6 Orientation

## 6.1 Introduction

The orientation of an object in space specifies how that object is aligned with respect to a reference configuration of that object. The reference configuration is a conceptual copy of the object that is defined with respect to a particular spatial reference frame. The orientation of the object may be specified by a distance preserving transformation that would make the reference configuration congruent to the object. Only the rotational components of this transformation are essential for the specification, as translation operations do not affect alignment.

For computational purposes, an orthonormal set of axes is created and attached to the object. These axes are termed the *object axes*. Another orthonormal set of axes is created and attached in the same manner to the corresponding position of the reference configuration. These axes are termed the *reference axes*. An orientation specification is a rotation operation that would bring the reference axes into alignment with the corresponding object axes. (An alternative method of specification defines the inverse rotation operation, that is, the rotation that would bring the object axes into alignment with the reference axes.) Only a single rotation is required for such a specification, since, as a consequence of Euler's rotation theorem, a given series of rotations is equivalent to a single rotation.

Rotation operator concepts and various mathematical representations of rotations have been in wide use from before the time of Euler's work on the subject. As a result, there are many different treatments in the literature, using similar terms with different meanings and different notational conventions. For this reason, rotation terms and notation used in this International Standard are fully defined.

The specification of an ORM (see 7.4.4) depends on a similarity transformation (see 7.3.2) for which a rotation operator is a key component. Converting the representation of such rotation operators to and from the Matrix representation (see 6.4.2) is required for some change of SRF operations (see 10.3.2 and 10.4.5). Rotation operators are also important in some of the application domains that fall within the scope of this International Standard. This includes the ability to convert an object's orientation represented with respect to one SRF to its equivalent with respect to another SRF.

## 6.2 Rotation operations and orientation

*Euler's rotation theorem* states that any distance-preserving transformation of 3D space that has at least one point fixed under the transformation is equivalent to a single rotation about an axis through an angle of rotation. If the axis is assigned a direction, the angle of rotation can be specified as a positive angle or a negative angle using the *right-hand rule*: conceptually, if the right-hand holds the axis with thumb pointing in the direction of the vector, the fingers curl in the positive angle direction.

There are two conventions in use for specifying the angle of rotation. Either the angle is measured from the starting position of a point to its rotated position, or it is measured from its rotated position to its starting position. The first convention is the *position vector rotation* (PVR) convention, and the second convention is the *coordinate frame rotation* (CFR) convention. Figure 6.1 illustrates the two conventions for a point $\boldsymbol{r}$ that is rotated to a new position $\boldsymbol{r}'$ about an axis that is perpendicular to the plane of the figure.

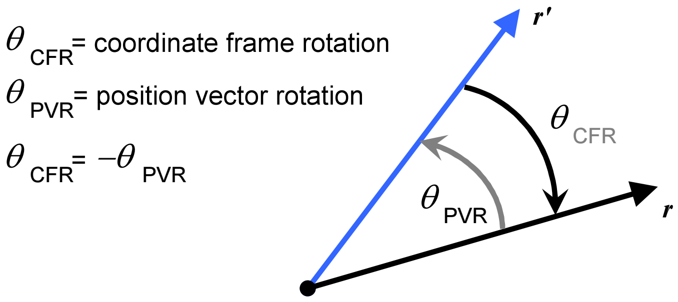


**Figure 6.1 — Rotation between *r* and *r′* in two conventions**

If $\boldsymbol{n}$ is a unit vector spanning a directed rotation axis, $\boldsymbol{R}_n(\theta)$ shall denote the rotation about the axis $\boldsymbol{n}$ through angle $\theta$ in PVR convention, and $\boldsymbol{\Omega}_n(\theta)$ shall denote the rotation about the axis $\boldsymbol{n}$ through angle $\theta$ in the CFR convention. The two conventions are related as follows: $\boldsymbol{R}_{-\boldsymbol{n}}(\theta) = \boldsymbol{\Omega}_{\boldsymbol{n}}(\theta) = \boldsymbol{R}_{\boldsymbol{n}}(-\theta)$. An *orientation specification* for an object with respect to a reference shall be specified by either:

a) A rotation $\boldsymbol{R}_{\boldsymbol{n}}(\theta)$ in PVR convention that would rotate the object from the reference configuration to align with the object configuration, or

b) A rotation $\boldsymbol{\Omega}_{\boldsymbol{n}}(\theta)$ in CFR convention that would rotate the object from the object configuration to align with the reference configuration.

The relationship of rotation operations (in a given rotation convention) and orientation specifications are closely related, but is not one-to-one. The rotations $\boldsymbol{R}_n(\theta + 2\pi k)$, where $k$ is any positive or negative integer value, are distinct rotations that all correspond to the same orientation specification. Thus only the angle of rotation modulo $2\pi$ determines orientation. The same holds for CFR convention $\boldsymbol{\Omega}_n(\theta + 2\pi k)$. Large rotations (greater than one full revolution) are important in some applications, however, in this International Standard angles shall be considered equivalent modulo $2\pi$.

Two consecutive rotations result in a composite transformation that is also a rotation. In the PVR convention, if a rotation $\boldsymbol{R}_n(\theta)$ is followed by a second rotation $\boldsymbol{R}_m(\varphi)$, the composite rotation in right-to-left operator order is $\boldsymbol{R}_m(\varphi) \circ \boldsymbol{R}_n(\theta)$. The composite rotation in the coordinate frame convention reverses the operator order:

$$\boldsymbol{\Omega}_n(-\theta) \circ \boldsymbol{\Omega}_m(-\varphi) = \boldsymbol{R}_m(\varphi) \circ \boldsymbol{R}_n(\theta). \tag{6.1}$$

If the orientation of an object $\boldsymbol{E}$ with respect to a reference $\boldsymbol{E}_0$ is specified in PVR convention by $\boldsymbol{R}_{E_0 E}$ (or in CFR convention $\boldsymbol{\Omega}_{E_0 E}$) and if $\boldsymbol{F}_0$ is another reference and the orientation of $\boldsymbol{E}_0$ with respect to $\boldsymbol{F}_0$ is known, the orientation of $\boldsymbol{E}$ with respect to $\boldsymbol{F}_0$ is computed (in right-to-left operator order) as:

$$\begin{aligned} \boldsymbol{R}_{F_0 E} &= \boldsymbol{R}_{E_0 E} \circ \boldsymbol{R}_{F_0 E_0} && \text{PVR convention} \\ \boldsymbol{\Omega}_{F_0 E} &= \boldsymbol{\Omega}_{F_0 E_0} \circ \boldsymbol{\Omega}_{E_0 E} && \text{CFR convention} \end{aligned} \tag{6.2}$$

NOTE The order of the rotations is important because rotation operators are not commutative.

## 6.3 Rodrigues' rotation formula

The notion of a rotation about an axis though a given rotation angle is independent of any selection of a Euclidean coordinate system (i.e., coordinate free). If a rotation operator $\boldsymbol{R}_{\boldsymbol{n}}(\theta)$ in PVR convention rotates point $\boldsymbol{r}$, the resulting rotated point $\boldsymbol{r}'$ may be computed using (coordinate free) vector space operations using Rodrigues' rotation formula (see [BERN]):

$$\boldsymbol{r}' = \cos(\theta)\boldsymbol{r} + (1-\cos(\theta))(\boldsymbol{r} \bullet \boldsymbol{n})\boldsymbol{n} + \sin(\theta)\boldsymbol{n}\times\boldsymbol{r} \tag{6.3}$$

The terms may be rearranged to the alternate form:

$$\boldsymbol{r}' = \boldsymbol{r} + (1-\cos(\theta))\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{r}) + \sin(\theta)\boldsymbol{n}\times\boldsymbol{r} \tag{6.4}$$

This formulation also applies to the CFR convention operator $\boldsymbol{\Omega}_{\boldsymbol{n}}(-\theta)$.

## 6.4 Representations of Rotations

### 6.4.1 Representation degrees of freedom and computational complexity

A consequence of Euler's rotation theorem is that any rotation operation on 3D Euclidean space has three degrees of freedom and may be specified by three scalar numbers. That is explicitly the case with Euler angle conventions (see 6.4.4.2).

Other less compact specifications using four or more scalars together with constraint rules are commonly used because they are more amenable to some computations such as performing a rotation operation on a point, composing rotations, interpolating rotations, and other operations and/or because these parameters can be measured or modelled directly. The Matrix representation (see 6.4.2) and the Quaternion representation (see 6.4.5) are in common use because the rotation of a point and the composition of rotations are directly computable as matrix or quaternion multiplications. Computing the composition of rotations in the Axis-angle representation (see 6.4.3) or in an Euler angle convention (see 6.4.4) usually require conversion to and from Matrix or Quaternion forms. All rotation representations defined below tacitly require an orthonormal basis for the coordinate representation of vectors.

The various representation methods in prevalent use present different tradeoffs with respect to storage size, computational complexity, speed, and error control. Thus the best representation is dependent on the requirements and computational environment of a user application. For this reason, different representations are in use and interoperability becomes an issue. This issue is compounded by the non-standard meaning of terms in prevalent use. To support interoperability and SRM operations, this International Standard defines these terms and identifies several representation methods as well as algorithms for key operations on and inter-conversions between the representation methods.

### 6.4.2 Matrix representation

A 3x3 matrix $\boldsymbol{M}$ is a rotation matrix, if it satisfies these properties:

$$\begin{aligned} \det(\boldsymbol{M}) &= 1 \\ \boldsymbol{M}^{\mathrm{T}} &= \boldsymbol{M}^{-1} \end{aligned} \tag{6.5}$$

Matrices satisfying these properties form an algebraic group with respect to matrix multiplication. This group is known as the *special orthogonal group* of degree 3, SO(3). In particular, the product of any two rotation matrices is itself a rotation matrix.

As a consequence of Euler's rotation theorem, the matrix has a unit eigenvector $\boldsymbol{n}$ and three eigenvalues: $1, e^{+i\theta}, e^{-i\theta}$. The transformation is then a rotation of positive angle $\theta$ about the rotation axis spanned by the vector $\boldsymbol{n}$ (the rotation axis points are fixed points under the transformation).

If $\boldsymbol{M} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, a corresponding axis unit vector $\boldsymbol{n}$ and angle of rotation $\theta$ is algorithmically determined as follows:

$$\theta = \arccos\left(\left(\frac{\mathrm{Trace}(\boldsymbol{M})-1}{2}\right)\right) = \arccos\left(\left(\frac{(a_{11}+a_{22}+a_{33})-1}{2}\right)\right),\ 0 \le \theta \le \pi.$$

There are three cases for the computation of $\boldsymbol{n}$ that depend on the value of $\theta$.

Case $\theta = 0$: There is no rotation so $\boldsymbol{n}$ is indeterminate.

Case $0 < \theta < \pi$: Let $\boldsymbol{n} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$, where:

$\boldsymbol{v} = \begin{pmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{pmatrix}$. In this case, $\|v\| = 2\left|\sin(\theta)\right|$.

Case: $\theta = \pi$: First find the maximum diagonal element $a_{11}$, $a_{22}$, or $a_{33}$ of $\boldsymbol{R}$. Then:

Sub-case: $a_{11}$ is the maximum and $\boldsymbol{v} = (a_{11}+1,\ a_{12},\ a_{13})^{\mathrm{T}}$.

Sub-case: $a_{22}$ is the maximum and $\boldsymbol{v} = (a_{21},\ a_{22}+1,\ a_{23})^{\mathrm{T}}$.

Sub-case: $a_{33}$ is the maximum and $\boldsymbol{v} = (a_{31},\ a_{32},\ a_{33}+1)^{\mathrm{T}}$.

Finally $\boldsymbol{n} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$.

In all cases, $-\boldsymbol{n}$ and $-\theta$ is also a solution.

The matrix $\boldsymbol{M}$ operates on 3D Euclidean space by either right or left matrix multiplication of vectors. The left multiply operation $\boldsymbol{r}' = \boldsymbol{M}\,\boldsymbol{r}$ corresponds to the PVR convention $\boldsymbol{R}_{\boldsymbol{n}}(\theta)$. The right multiply operation $\boldsymbol{r}' = \boldsymbol{r}\,\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}}\,\boldsymbol{r}$ corresponds to the CFR convention $\boldsymbol{\Omega}_{\boldsymbol{n}}(\theta)$. The product of two rotation matrices corresponds to the composition of the two rotations.

NOTE 1 Matrix multiplication is generally not commutative.

NOTE 2 The matrix has nine parameters; however the constraints on the determinant and the transpose reduce the degrees of freedom to three.

A special case of a rotation matrix arises from a change of basis operation. If $\boldsymbol{r}$ is a point in 3D Euclidean space and $\boldsymbol{E}$ denotes that vector space with orthonormal basis $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, and $\boldsymbol{E}'$ denotes the same vector space with orthonormal basis $\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}'$. The coordinate representation of $\boldsymbol{r}$ with respect to each basis is:

$\boldsymbol{r} = (r_1 \quad r_2 \quad r_3)^{\mathrm{T}}$, where $\boldsymbol{r} = r_1\boldsymbol{x} + r_2\boldsymbol{y} + r_3\boldsymbol{z}$, and

$\boldsymbol{r} = (r_1' \quad r_2' \quad r_3')^{\mathrm{T}}$, where $\boldsymbol{r} = r_1'\boldsymbol{x}' + r_2'\boldsymbol{y}' + r_3'\boldsymbol{z}'$.

The matrix $\boldsymbol{M}$ corresponding to the coordinate basis transformation from $\boldsymbol{E}$ to $\boldsymbol{E}'$ : $(r_1, r_2, r_3) \mapsto (r_1', r_2', r_3')$ is given by the *direction cosine matrix*:

$$\boldsymbol{M} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x} \bullet \boldsymbol{x}' & \boldsymbol{y} \bullet \boldsymbol{x}' & \boldsymbol{z} \bullet \boldsymbol{x}' \\ \boldsymbol{x} \bullet \boldsymbol{y}' & \boldsymbol{y} \bullet \boldsymbol{y}' & \boldsymbol{z} \bullet \boldsymbol{y}' \\ \boldsymbol{x} \bullet \boldsymbol{z}' & \boldsymbol{y} \bullet \boldsymbol{z}' & \boldsymbol{z} \bullet \boldsymbol{z}' \end{pmatrix} \tag{6.6}$$

NOTE 3 The direction cosine matrix is so named because each dot product in Equation (6.6) is the cosine of the angle between the two indicated unit vectors.

The columns of the matrix are the $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ basis vectors in $\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}'$ coordinate representation while the rows (or columns of the transpose matrix) are the $\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}'$ basis vectors in $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ coordinate representation. Thus the transpose of the matrix corresponds to the inverse transformation $(r_1', r_2', r_3') \mapsto (r_1, r_2, r_3)$.

### 6.4.3 Axis-angle representation

The *axis-angle* representation $(\boldsymbol{n}, \theta)$ is a coordinate system dependent representation of a PVR convention rotation $\boldsymbol{R}_{\boldsymbol{n}}(\theta)$. It consists of a unit vector $\boldsymbol{n} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^{\mathrm{T}}$ and a rotation angle $\theta$. The corresponding rotation in the CFR convention is $\boldsymbol{\Omega}_{\boldsymbol{n}}(-\theta) = \boldsymbol{\Omega}_{-\boldsymbol{n}}(\theta)$. This representation uses four scalar parameters $n_1, n_2, n_3$ and $\theta$. The unit constraint $\|\boldsymbol{n}\| = 1$ reduces the degrees of freedom to three. The axis-angle representation is not unique. In particular, the axis-angle pairs $(\boldsymbol{n}, \theta)$ and $(-\boldsymbol{n}, -\theta)$ represent the same rotation, and when $\theta = 0$, $\boldsymbol{n}$ may be any unit vector or the zero vector.

NOTE A three parameter version in the form $(a_1, a_2, a_3) = (\theta \boldsymbol{n})$ is also in use. In this form, $\theta$ is non-negative and is computed as $\theta = \|(a_1, a_2, a_3)\|$ and $\boldsymbol{n} = \frac{1}{\theta}(a_1, a_2, a_3)$ when $\theta \neq 0$.

The operation of an axis-angle rotation $(\boldsymbol{n}, \theta)$ on 3D Euclidean space is given by Rodrigues' rotation formula (Equation (6.3)). There is no direct computational formulation of the composition of two axis-angle rotations in axis-angle form.

### 6.4.4 Principal rotations and Euler angle conventions

#### 6.4.4.1 Principal rotations

Principal rotations depend on a given orthonormal basis for 3D Euclidean space. Unit axis vectors may be represented in that basis by the coordinate 3-tuples: $\boldsymbol{x} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$, $\boldsymbol{y} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^{\mathrm{T}}$, and $\boldsymbol{z} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}}$. As an axis of rotation, each of these unit vectors is termed a *principal axis* of rotation. A rotation about a principal axis is termed a *principal rotation*. Some authors refer to these rotations as *elementary rotations*. The vector space operators: $\boldsymbol{R}_x(\alpha), \boldsymbol{R}_y(\beta)$, and $\boldsymbol{R}_z(\gamma)$ denote the three principal rotations through the respective angles $\alpha, \beta$, and $\gamma$ modulo $2\pi$ in the PVR convention. Principal rotations in the CFR convention are denoted by $\boldsymbol{\Omega}_x(\alpha)$, $\boldsymbol{\Omega}_y(\beta)$, and $\boldsymbol{\Omega}_z(\gamma)$.

#### 6.4.4.2 Euler angles

*Euler angles* are a specification of a rotation obtained by applying three consecutive principal rotations. There are twelve distinct ways to select a sequence of three principal axes and apply the principal position rotations

 

(24 if left-handed axes are considered)[19]. Each such ordered selection of axes is an *Euler angle convention*. There is little agreement among authors on names or notations for these conventions. There are numerous Euler angle conventions in use and many are named inconsistently. Some authors use a left-handed coordinate system. All coordinate systems in this International Standard are right-handed.

This International Standard adopts the following convention and notation for Euler angles: Given a 3-tuple of Euler angles $(\alpha,\beta,\gamma)$ the Euler convention specification shall be specified by a character string denoting the sequence of principal axes in the form $A_1 - A_2 - A_3$ where each symbol $A_1$, $A_2$, $A_3$ is one of the axis letters $x$, $y$, or $z$. Thus $(\alpha,\beta,\gamma)$ in the $z$-$x$-$z$ Euler convention is the composite of a principal rotation about the $z$-axis first, the $x$-axis second, and the $z$-axis again for the third rotation.

The three angles representing a rotation in a given Euler angle convention are not necessarily unique modulo $2\pi$. The conditions that result in non-unique angle 3-tuples are given in Table 6.4 for the $\boldsymbol{z}$-$\boldsymbol{x}$-$\boldsymbol{z}$ Euler angle convention and in Table 6.6 for the $\boldsymbol{x}$-$\boldsymbol{y}$-$\boldsymbol{z}$ Euler angle convention (see also 6.4.4.5).

There are several ways to realize an Euler angle sequence. In the PVR convention, the three principal rotations may either be rotations about the original axes, or about the successively rotated axes. Given a rotation, let $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}$ denote the principal axes after the successive rotations are applied to the original $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ axes. To distinguish between these two coordinate bases, coordinates with respect to the (static) original basis $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ shall be termed *space-fixed* coordinates and those with respect to the sequentially rotating $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}$ axes shall be termed *body-fixed* coordinates. It is useful to think of the $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}$ as attached to a rigid entity that will be rotated. In the CFR convention, the realization is similar to the PVR body-fixed case in that the rotation angles are measured from the rotated axes (see Equation (6.1)). These three realizations of $(\alpha,\beta,\gamma)$ in the $A_1 - A_2 - A_3$ Euler convention (in right-to-left operator order) are:

$$\begin{aligned}
&\boldsymbol{R}_{A_3}(\gamma)\circ\boldsymbol{R}_{A_2}(\beta)\circ\boldsymbol{R}_{A_1}(\alpha) && \text{PVR convention} \quad \text{space-fixed}\\
&=\boldsymbol{R}_{\tilde{A}_1}(\alpha)\circ\boldsymbol{R}_{\tilde{A}_2}(\beta)\circ\boldsymbol{R}_{\tilde{A}_3}(\gamma) && \text{PVR convention} \quad \text{body-fixed}\\
&=\boldsymbol{\Omega}_{A_1}(-\alpha)\circ\boldsymbol{\Omega}_{A_2}(-\beta)\circ\boldsymbol{\Omega}_{A_3}(-\gamma) && \text{CFR convention}
\end{aligned} \tag{6.7}$$

In Equation (6.7) axis letters A denote static axes and letters Ã denote successively rotated axes.

EXAMPLE 1 Using Equation (6.7), $(\pi/6, \pi/4, \pi/2)$ in the $\boldsymbol{z}$-$\boldsymbol{x}$-$\boldsymbol{z}$ Euler convention, the PVR realizations are: $\boldsymbol{R}_z(\pi/2)\circ\boldsymbol{R}_x(\pi/4)\circ\boldsymbol{R}_z(\pi/6)$ space-fixed, and $\boldsymbol{R}_{\tilde{z}}(\pi/6)\circ\boldsymbol{R}_{\tilde{x}}(\pi/4)\circ\boldsymbol{R}_{\tilde{z}}(\pi/2)$ body-fixed.

EXAMPLE 2 Substituting $(\psi,\theta,\varphi)$ in Equation (6.7), the Euler $\boldsymbol{x}$-$\boldsymbol{y}$-$\boldsymbol{z}$ convention has the following PVR realizations: $\boldsymbol{R}_z(\varphi)\circ\boldsymbol{R}_y(\theta)\circ\boldsymbol{R}_x(\psi)$ space-fixed, and $\boldsymbol{R}_{\tilde{x}}(\psi)\circ\boldsymbol{R}_{\tilde{y}}(\theta)\circ\boldsymbol{R}_{\tilde{z}}(\varphi)$ body-fixed.

There are no direct computational formulations for the operation of an Euler angle rotation on 3D Euclidean space or for representing the composition of two Euler angle rotations as a single Euler angle rotation. For these computations, the principal rotation sequence is commonly realized as a product of matrices or quaternions.

[19] There cannot be two consecutive rotations on the same axis as they would combine to a single rotation. Thus, among right-handed axis systems, there are 3 choices for the first rotation axis, 2 choices each for the second and third rotation axes to avoid repeating the preceding axis choice (3x2x2=12).

 

#### 6.4.4.3 The $z$-$x$-$z$ convention

Generally, the initial $xy$-plane and the final rotated $\tilde{x}\tilde{y}$-plane intersect in a line. This line is termed the *line of nodes* for this convention. The Euler angles in the $z$-$x$-$z$ convention are the three angles defined as follows:

$\alpha$ is the angle between the line of nodes and the $\tilde{x}$-axis,
$\beta$ is the angle between the $z$-axis and the $\tilde{z}$-axis, and
$\gamma$ is the angle between the $x$-axis and the line of nodes.

In the case that the initial $xy$-plane lies in the final rotated $\tilde{x}\tilde{y}$-plane, $\beta = 0$ or $\beta = \pi$ (see 6.4.4.5).

In some contexts $\alpha, \beta, \gamma$ are known, respectively, as the *spin* angle, the *nutation* angle, and the *precession* angle. These three angles specify a rotation as consecutive principal rotations using the $z$–axis, the $x$–axis and the $z$–axis again. The three realizations (in right-to-left operator order) are:

$\boldsymbol{R}_z(\gamma) \circ \boldsymbol{R}_x(\beta) \circ \boldsymbol{R}_z(\alpha)$ PVR convention space-fixed,
$\boldsymbol{R}_{\tilde{z}}(\alpha) \circ \boldsymbol{R}_{\tilde{x}}(\beta) \circ \boldsymbol{R}_{\tilde{z}}(\gamma)$ PVR convention body-fixed, and
$\boldsymbol{\Omega}_z(-\alpha) \circ \boldsymbol{\Omega}_x(-\beta) \circ \boldsymbol{\Omega}_z(-\gamma)$ CFR convention.

In the PVR body-fixed realization the first principal rotation is about the $z$-axis through angle $\alpha$, which rotates the $x$-axis to intermediate position $x'$. Next is a rotation about the $x'$-axis through angle $\beta$, which rotates the $z$-axis to $z''$. The third rotation is about the $z''$-axis through angle $\gamma$. The sequence of PVR body-fixed rotations is illustrated in Figure 6.2.

#### 6.4.4.4 The $x$-$y$-$z$ convention and Tait-Bryan angles

In this convention the line of nodes is the intersection of the $xy$-plane and the final rotated $\tilde{y}\tilde{z}$-plane. The Euler angles in this convention are defined as follows:

$\phi$ is the angle between the line of nodes and the $\tilde{y}$-axis,
$\theta$ is the angle between $\tilde{x}$-axis and the $xy$-plane, (equivalently, the $z$-axis and the $\tilde{y}\tilde{z}$-plane), and
$\psi$ is the angle between the $y$-axis and the line of nodes.

These three angles $(\varphi, \theta, \psi)$ specify a rotation that may be realized (in right-to-left operator order) as:

$\boldsymbol{R}_z(\psi) \circ \boldsymbol{R}_y(\theta) \circ \boldsymbol{R}_x(\phi)$ PVR convention space fixed,
$\boldsymbol{R}_{\tilde{x}}(\phi) \circ \boldsymbol{R}_{\tilde{y}}(\theta) \circ \boldsymbol{R}_{\tilde{z}}(\psi)$ PVR convention body-fixed and
$\boldsymbol{\Omega}_x(-\phi) \circ \boldsymbol{\Omega}_y(-\theta) \circ \boldsymbol{\Omega}_z(-\psi)$ CFR convention.

 

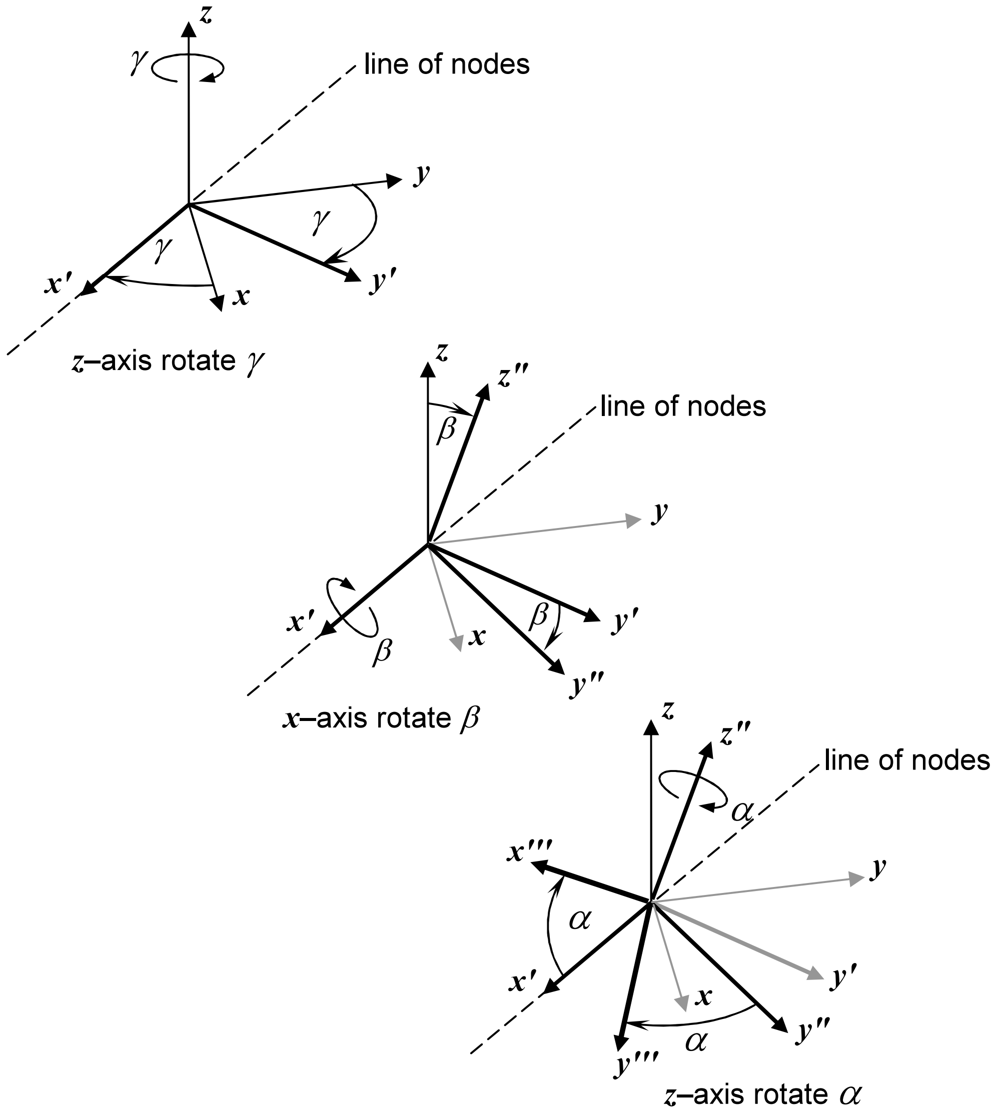


**Figure 6.2 — Euler *z-x-z* body-fixed realization**

The Euler angles in the PVR body-fixed realization are variously termed *Tait-Bryan angles*, *Cardano angles*, or *nautical angles*. The various names given to these angle symbols include:

$\phi$ roll or bank or tilt,
$\theta$ pitch or elevation, and
$\psi$ yaw or heading or azimuth (see Figure 6.3).

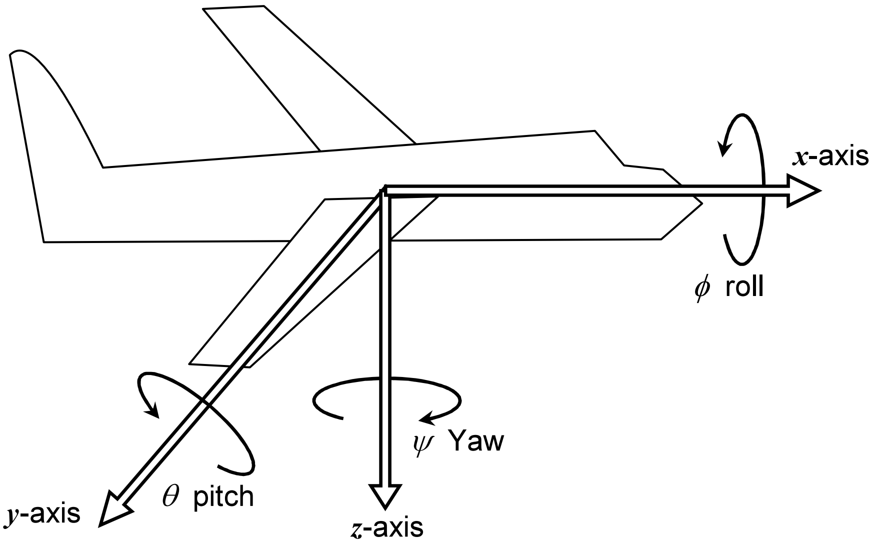


**Figure 6.3 — Tait-Bryan angles**

#### 6.4.4.5 Gimbal lock

The term *gimbal lock* refers to a gyroscope mounted in three nested gimbals to provide three degrees of rotational freedom. Each mounting scheme corresponds to an Euler angle convention. In any such mounting scheme, there exist critical angles for the middle gimbal that reduce the rotational degrees of freedom from three to two. In those critical configurations, the gimbals lie in a single plane and rotation within that plane is figuratively "locked out" by the gimbal mechanism. This loss of a degree of freedom is termed "gimbal lock".

In the case of the Euler angle $z$-$x$-$z$ rotation convention, it is assumed that the $\boldsymbol{xy}$-plane and $\tilde{x}\tilde{y}$-plane intersect in a line (the line on nodes). That assumption is met when (modulo 2π) $\beta \neq 0$ and $\beta \neq \pi$. If not, $\beta = 0$ or $\beta = \pi$ and the consecutive rotations collapse down to a single principal rotation:

$$\begin{aligned} \beta = 0: &\quad \boldsymbol{R}_z(\gamma) \circ \boldsymbol{R}_x(0) \circ \boldsymbol{R}_z(\alpha) = \boldsymbol{R}_z(\gamma) \circ \boldsymbol{R}_z(\ \alpha) = \boldsymbol{R}_z(\gamma + \alpha) \\ \beta = \pi: &\quad \boldsymbol{R}_z(\gamma) \circ \boldsymbol{R}_x(\pi) \circ \boldsymbol{R}_z(\alpha) = \boldsymbol{R}_z(\gamma) \circ \boldsymbol{R}_z(-\alpha) = \boldsymbol{R}_z(\gamma - \alpha) \end{aligned}. \tag{6.8}$$

NOTE 1 This situation is illustrated by a spinning table top. The top spins on its spin-axis and precesses about the precession-axis. The angle between the spin- and precession-axes is the nutation angle. When the spin-axis is perfectly vertical (either upright or upside down), the nutation angle is 0 or π and the spin- and precession-axes become indistinguishable from each other as indicated in Equation (6.8).

In the case of the Euler angle $x$-$y$-$z$ convention (Tait-Bryan angles) it is assumed that the $\boldsymbol{xy}$-plane and $\tilde{y}\tilde{z}$-plane intersect in a line (the line of nodes). That assumption is met when $\theta \neq \pm\pi/2$ modulo 2π. If not, $\theta = \pm\pi/2$ and the $\tilde{x}$-axis becomes parallel to the $z$-axis and the consecutive rotations collapse down to a single principal rotation:

$$\begin{aligned} \theta = +\pi/2: &\quad \boldsymbol{R}_z(\psi) \circ \boldsymbol{R}_y(\pi/2) \circ \boldsymbol{R}_x(\phi) = \boldsymbol{R}_z(\psi + \phi) \\ \theta = -\pi/2: &\quad \boldsymbol{R}_z(\psi) \circ \boldsymbol{R}_y(-\pi/2) \circ \boldsymbol{R}_x(\phi) = \boldsymbol{R}_z(\psi - \phi) \end{aligned}. \tag{6.9}$$

NOTE 2 This situation is illustrated by an aircraft as in Figure 6.3. When the aircraft either climbs vertically, or dives vertically, roll-rotation cannot be distinguished from (plus or minus) yaw-rotation. This occurs at critical pitch angles of $\theta = \pm\pi/2$ as indicated in Equation (6.9).

### 6.4.5 Quaternion representation

#### 6.4.5.1 Quaternion notations and conventions

The quaternion system is a 4-dimensional vector space together with a vector multiplication operation that forms a non-commutative associative algebra. In analogy to complex numbers that are written as $a + \boldsymbol{i}b,\ \boldsymbol{i}^2 = -1$, quaternion axes $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, are defined with the following relationships: $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. There are several notational conventions in use including the three termed in this International Standard as the *Hamilton form*, the *4-tuple form*, and the *scalar vector form*. In these notation forms a quaternion $\boldsymbol{q}$ is denoted as follows:

| | |
|---|---|
| $\boldsymbol{q} = e_0 + e_1\boldsymbol{i} + e_2\boldsymbol{j} + e_3\boldsymbol{k}$ | Hamilton form |
| $\boldsymbol{q} = (e_0, e_1, e_2, e_3)$ | 4-tuple form |
| $\boldsymbol{q} = (e_0, \boldsymbol{e}),\ \boldsymbol{e} = (e_1 \quad e_2 \quad e_3)^{\mathrm{T}}$ | scalar vector form |

where $e_0, e_1, e_2, e_3$ are scalar values.

The $e_0$ value is termed the *real* (or “scalar”) part of $\boldsymbol{q}$ and $(e_1, e_2, e_3)$ is termed the *imaginary* (or “vector”) part of $\boldsymbol{q}$. The remainder of this clause uses the scalar vector form.

NOTE 1 In the literature, the component order of the scalar vector form is sometimes reversed: $\boldsymbol{q} = (\boldsymbol{e}, e_0)$.

NOTE 2 A unit quaternion (see below) in 4-tuple form is also termed the *Euler parameters* (or the *Euler-Rodrigues parameters*) of a rotation. In the literature, the real part of the 4-tuple form is sometimes placed last: $\boldsymbol{q} = (e_1, e_2, e_3, e_4)$ where $e_4 = e_0$.

#### 6.4.5.2 Quaternion algebra

Quaternion multiplication and other operations are defined in Annex A in all the three notational forms. Given quaternions $\boldsymbol{q} = (e_0, \boldsymbol{e})$ and $\boldsymbol{p} = (d_0, \boldsymbol{d})$, A.10 defines:

the product $\boldsymbol{pq} = \left((d_0 e_0 - \boldsymbol{d} \bullet \boldsymbol{e}), (e_0\boldsymbol{d} + d_0\boldsymbol{e} + \boldsymbol{d} \times \boldsymbol{e})\right)$,

the conjugate $\boldsymbol{q}^* = (e_0, -\boldsymbol{e})$,

the norm $\boldsymbol{qq}^* = \boldsymbol{q}^*\boldsymbol{q} = \left(e_0^2 + e_1^2 + e_2^2 + e_3^2,\ \boldsymbol{0}\right)$, and

the modulus $|\boldsymbol{q}| = \sqrt{\boldsymbol{qq}^*} = \sqrt{e_0^2 + e_1^2 + e_2^2 + e_3^2}$.

A quaternion $\boldsymbol{q}$ is a *unit quaternion* if $|\boldsymbol{q}| = 1$. In that case $\boldsymbol{qq}^* = \boldsymbol{q}^*\boldsymbol{q} = (1, \boldsymbol{0})$ which is the multiplicative identity so that, for a unit quaternion, its conjugate is its multiplicative inverse $\boldsymbol{q}^{-1} = \boldsymbol{q}^*$. Any unit quaternion may be expressed in the form:

$$q = \left(\cos(\theta/2), \sin(\theta/2)\,\boldsymbol{n}\right) \tag{6.10}$$

where:

$\boldsymbol{n} = \frac{1}{\|\boldsymbol{e}\|}\boldsymbol{e}$ is a unit vector in 3D space,

$\theta = 2 \cdot \mathrm{arctan2}\left(\sqrt{e_1^2 + e_2^2 + e_3^2},\, e_0\right)$.

NOTE The two argument arctangent function arctan2() is defined in Annex A.

#### 6.4.5.3 Quaternion operators on 3D Euclidean space

Each quaternion $\boldsymbol{q}$ corresponds to a transformation of 3D Euclidean space as follows. If $\boldsymbol{r}$ is a vector in 3D Euclidean space, the corresponding quaternion is formed by using 0 for the real part and $\boldsymbol{r}$ for the imaginary part $(0,\boldsymbol{r})$. A unit quaternion $\boldsymbol{q}$ operates on $(0,\boldsymbol{r})$ by left multiplying with $\boldsymbol{q}$ and right multiplying with its conjugate $\boldsymbol{q}^*$. The real part of the product $\boldsymbol{q}(0,\boldsymbol{r})\boldsymbol{q}^* = (r_0', \boldsymbol{r}')$, is 0. Thus, $\boldsymbol{q}(0,\boldsymbol{r})\boldsymbol{q}^* = (0,\boldsymbol{r}')$ is pure imaginary and the quaternion $\boldsymbol{q}$ associates $\boldsymbol{r}'$ with $\boldsymbol{r}$. Symbolically the operation on $\boldsymbol{r}$ is:

$$\boldsymbol{r} \mapsto \boldsymbol{r}' = \mathit{imaginary\ part}\left\{\boldsymbol{q}(0,\boldsymbol{r})\boldsymbol{q}^*\right\}. \tag{6.11}$$

This is equivalent to:

$$\boldsymbol{r}' = \left(e_0^2 - \boldsymbol{e} \bullet \boldsymbol{e}\right)\boldsymbol{r} + 2\left(\boldsymbol{e} \bullet \boldsymbol{r}\right)\boldsymbol{e} + 2e_0\boldsymbol{e} \times \boldsymbol{r}. \tag{6.12}$$

$-\boldsymbol{q} = (-e_0, -\boldsymbol{e})$ produces the same $\boldsymbol{r}'$ so that $\boldsymbol{q}$ and $-\boldsymbol{q}$ produce equivalent rotations.

If $\boldsymbol{q} = \left(\cos(\theta/2), \sin(\theta/2)\,\boldsymbol{n}\right)$ is a unit quaternion, Equation (6.12) reduces to the Rodrigues rotation formula for a clockwise rotation about $\boldsymbol{n}$ through angle $\theta$:

$$\boldsymbol{r}' = \cos(\theta)\boldsymbol{r} + \left(1 - \cos(\theta)\right)\left(\boldsymbol{n} \bullet \boldsymbol{r}\right)\boldsymbol{n} + \sin(\theta)\boldsymbol{n} \times \boldsymbol{r}.$$

A non-zero quaternion $\boldsymbol{p}$ and its corresponding unit quaternion $\boldsymbol{q} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|^2}$ perform the same rotation

$$\boldsymbol{p}(0,\boldsymbol{r})\boldsymbol{p}^{-1} = \boldsymbol{q}(0,\boldsymbol{r})\boldsymbol{q}^*.$$

For this reason, some authors use $\boldsymbol{p}(0,\boldsymbol{r})\boldsymbol{p}^{-1}$ operations for any non-zero quaternion while others use the $\boldsymbol{q}(0,\boldsymbol{r})\boldsymbol{q}^*$ operator and restrict operations only to unit quaternions.

The quaternion representation of rotation facilitates the computation of the composition of two rotations.

If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are two unit quaternions, the composite rotation on $\boldsymbol{r}$ that is obtained by first rotating with the rotation induced by $\boldsymbol{q}_1$ and then rotating the result with the rotation induced by $\boldsymbol{q}_2$ is the same as the single rotation induced by the product $\boldsymbol{q}_2\boldsymbol{q}_1$ since $\boldsymbol{q}_2\left\{\boldsymbol{q}_1(0,\boldsymbol{r})\boldsymbol{q}_1^*\right\}\boldsymbol{q}_2^* = \boldsymbol{q}_2\boldsymbol{q}_1(0,\boldsymbol{r})\boldsymbol{q}_1^*\boldsymbol{q}_2^* = \left\{\boldsymbol{q}_2\boldsymbol{q}_1\right\}(0,\boldsymbol{r})\left\{\boldsymbol{q}_2\boldsymbol{q}_1\right\}^*$.

### 6.4.6 Representation summary

Some important attributes of the representations in this section are summarized in Table 6.1.

**Table 6.1 — Summary of representation attributes**

| **Representation type** | **Data compo-nents** | **Data constraints** | **Ambiguities** (modulo $2\pi$) | **Composition** | **Inverse** |
|---|---|---|---|---|---|
| Axis-angle $(\boldsymbol{n}, \theta)$ | 4 | $\lVert\boldsymbol{n}\rVert = 1$ | $(\boldsymbol{n}, \theta)$ is equivalent to $(-\boldsymbol{n}, -\theta)$. If $\theta = 0$, $\boldsymbol{n}$ is indeterminate | Convert to/from another representation for the operation | $(\boldsymbol{n}, -\theta)$ or $(-\boldsymbol{n}, \theta)$ |
| Matrix $\boldsymbol{R}$ | 9 | $\det(\boldsymbol{R}) = 1$ $\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}^{-1}$ | None | Matrix multiplication | $\boldsymbol{R}^{\mathrm{T}}$ |
| Euler angle conventions | 3 | None | 2 or more<br>$z$-$x$-$z$ convention: see Table 6.4<br>Tait-Bryan $x$-$y$-$z$ angles: see Table 6.6 | Convert to/from another representation for the operation (see Note 2) | See Note 1 |
| Unit quaternion $\boldsymbol{q}$ | 4 | unit constraint: $\boldsymbol{q}\boldsymbol{q}^* = 1$ | $\boldsymbol{q}$ is equivalent to $-\boldsymbol{q}$ (see Note 3) | Quaternion multiplication | $\boldsymbol{q}^*$ or $-\boldsymbol{q}^*$ |

NOTE 1 The inverse in the Euler angle $z$-$x$-$z$ convention is

$$\left[\boldsymbol{R}_z(\gamma)\circ\boldsymbol{R}_x(\beta)\circ\boldsymbol{R}_z(\alpha)\right]^{-1} = \boldsymbol{R}_z(-\alpha)\circ\boldsymbol{R}_x(-\beta)\circ\boldsymbol{R}_z(-\gamma) = \boldsymbol{\Omega}_z(\alpha)\circ\boldsymbol{\Omega}_x(\beta)\circ\boldsymbol{\Omega}_z(\gamma)$$

The inverse in the Euler angle $z$-$y$-$x$ convention (Tait-Bryan angles) is

$$\left[\boldsymbol{R}_z(\psi)\circ\boldsymbol{R}_y(\theta)\circ\boldsymbol{R}_x(\phi)\right]^{-1} = \boldsymbol{R}_x(-\phi)\circ\boldsymbol{R}_y(-\theta)\circ\boldsymbol{R}_z(-\psi) = \boldsymbol{\Omega}_z(\psi)\circ\boldsymbol{\Omega}_y(\theta)\circ\boldsymbol{\Omega}_x(\phi)$$

NOTE 2 The composition of Euler angle operations may also be performed in a "direct" method that involves lengthy expressions combining forward and inverse trigonometric functions.

NOTE 3 Formulae such as Equation (6.12) require the unit quaternion constraint. Other useful relationships such as Equation (6.11) do not have that requirement. For that reason, some applications do not enforce the unit constraint. In the unconstrained case, every non-zero scalar multiple of a given quaternion is rotationally equivalent to it.

## 6.5 Inter-converting between rotations representations

### 6.5.1 Euler angle conventions and matrix representation

#### 6.5.1.1 Matrix forms of principal rotations

The matrix representations of principal rotations are given in Table 6.2.

**Table 6.2 — Principal rotations as matrix operators**

| Name | Notation | Matrix operator (left multiplication) |
|---|---|---|
| $x$-axis principal rotation CFR convention | $\boldsymbol{\Omega}_x(\omega_1)$ | $\boldsymbol{\Omega}_X(\omega_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega_1) & \sin(\omega_1) \\ 0 & -\sin(\omega_1) & \cos(\omega_1) \end{pmatrix}$, where $\omega_1$ is the angle of rotation. |
| $x$-axis principal rotation PVR convention | $\boldsymbol{R}_x(\omega_1)$ | $\boldsymbol{R}_X(\omega_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega_1) & -\sin(\omega_1) \\ 0 & \sin(\omega_1) & \cos(\omega_1) \end{pmatrix}$, where $\omega_1$ is the angle of rotation. |
| $y$-axis principal rotation CFR convention | $\boldsymbol{\Omega}_y(\omega_2)$ | $\boldsymbol{\Omega}_Y(\omega_2) = \begin{pmatrix} \cos(\omega_2) & 0 & -\sin(\omega_2) \\ 0 & 1 & 0 \\ \sin(\omega_2) & 0 & \cos(\omega_2) \end{pmatrix}$, where $\omega_2$ is the angle of rotation. |
| $y$-axis principal rotation PVR convention | $\boldsymbol{R}_y(\omega_2)$ | $\boldsymbol{R}_Y(\omega_2) = \begin{pmatrix} \cos(\omega_2) & 0 & \sin(\omega_2) \\ 0 & 1 & 0 \\ -\sin(\omega_2) & 0 & \cos(\omega_2) \end{pmatrix}$, where $\omega_2$ is the angle of rotation. |
| $z$-axis principal rotation CFR convention | $\boldsymbol{\Omega}_z(\omega_3)$ | $\boldsymbol{\Omega}_Z(\omega_3) = \begin{pmatrix} \cos(\omega_3) & \sin(\omega_3) & 0 \\ -\sin(\omega_3) & \cos(\omega_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $\omega_3$ is the angle of rotation. |
| $z$-axis principal rotation PVR convention | $\boldsymbol{R}_z(\omega_3)$ | $\boldsymbol{R}_Z(\omega_3) = \begin{pmatrix} \cos(\omega_3) & -\sin(\omega_3) & 0 \\ \sin(\omega_3) & \cos(\omega_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $\omega_3$ is the angle of rotation. |

#### 6.5.1.2 The $z$-$x$-$z$ Euler angle convention

The angle sequence $(\alpha, \beta, \gamma)$ in the Euler $z$-$x$-$z$ convention is converted to a matrix $\boldsymbol{M}$ by forming the matrix product of the corresponding three principal rotation matrices specified in Table 6.2. The resulting matrix is given in Equation (6.13).

$$\boldsymbol{M} = \boldsymbol{R}_z(\gamma) \circ \boldsymbol{R}_x(\beta) \circ \boldsymbol{R}_z(\alpha) = \begin{pmatrix} \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & -\sin\alpha\cos\gamma - \cos\beta\cos\alpha\sin\gamma & \sin\beta\sin\gamma \\ \cos\beta\sin\alpha\cos\gamma + \cos\alpha\sin\gamma & \cos\beta\cos\alpha\cos\gamma - \sin\alpha\sin\gamma & -\sin\beta\cos\gamma \\ \sin\beta\sin\alpha & \sin\beta\cos\alpha & \cos\beta \end{pmatrix} \tag{6.13}$$

Conversely, given a matrix $\boldsymbol{M}$ with elements $a_{ij}$, the equation may solved for the principal rotation factors $\boldsymbol{R}_z(\gamma) \circ \boldsymbol{R}_x(\beta) \circ \boldsymbol{R}_z(\alpha)$, and therefore solved for angles $(\alpha, \beta, \gamma)$. The solution is given in Table 6.3.

**Table 6.3 — Principal factors for the Euler *z*-*x*-*z* convention**

| Case | Principal factors for rotation $\boldsymbol{R}_z(\gamma) \circ \boldsymbol{R}_x(\beta) \circ \boldsymbol{R}_z(\alpha)$ (all angles modulo 2π) | | |
|---|---|---|---|
| $a_{33} \neq \pm 1$ | $\beta = \arccos(a_{33})$ [ principal value ] $0 < \beta < \pi$ | $\alpha = \text{arctan2}(a_{31}, a_{32})$ | $\gamma = \text{arctan2}(a_{13}, -a_{23})$ |
| | $\beta = \arccos(a_{33})$ [ $2\pi -$ principal value ] $\pi < \beta < 2\pi$ | $\alpha = \text{arctan2}(-a_{31}, -a_{32})$ | $\gamma = \text{arctan2}(-a_{13}, a_{23})$ |
| $a_{33} = -1$ | $\beta = \pi$ | any value of $\alpha$ | $\gamma = \text{arctan2}(a_{21}, a_{11}) + \alpha$ |
| $a_{33} = +1$ | $\beta = 0$ | any value of $\alpha$ | $\gamma = \text{arctan2}(a_{21}, a_{11}) - \alpha$ |

In the case $a_{33} \neq \pm 1$, arccos() is multi-valued so that there are two valid solution sets depending on the quadrants selected for arccosine values[20]. The principal value solution is the commonly used one. The two argument arctangent function arctan2() is defined in Annex A.

In the case $a_{33} = -1$, using trigonometric identities, the matrix expression reduces to :

$$\boldsymbol{R}_z(\gamma) \circ \boldsymbol{R}_y(\pi) \circ \boldsymbol{R}_z(\alpha) = \begin{pmatrix} \cos(\gamma - \alpha) & \sin(\gamma - \alpha) & 0 \\ \sin(\gamma - \alpha) & -\cos(\gamma - \alpha) & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For this reason, only the difference of the other two angles can be determined by using $\gamma - \alpha = \text{arctan2}(a_{21}, a_{11})$. Therefore, all values are valid for $\alpha$ if $\gamma = \text{arctan2}(a_{21}, a_{11}) + \alpha$. The case $a_{31} = +1$ is similar to the previous case with the sum of the angles determined by using $\gamma + \alpha = \text{arctan2}(a_{21}, a_{11})$. These two cases correspond to the gimbal lock Equation (6.8).

[20] Note that computer library functions such as acos() return the principal value only. The second solution for $\beta$ may be obtained by subtracting the principal value from $2\pi$.

As seen in the preceding tables, the three angle sequence corresponding to a given rotation or orientation operator is not unique modulo 2π. Two sequences, $(\alpha_1,\beta_1,\gamma_1)$ and $(\alpha_2,\beta_2,\gamma_2)$ of *z*-*x*-*z* principal factors specify the same operator if they satisfy one the criteria specified in Table 6.4.

**Table 6.4 — Equivalence of *z*-*x*-*z* principal factor sequences**

| **Case** (equality modulo 2π) | **Criteria for the equivalence of angle sequences $(\alpha_1,\beta_1,\gamma_1)$ and $(\alpha_2,\beta_2,\gamma_2)$ for principal factor *z*-*x*-*z* sequences** |
|---|---|
| $\beta_1 = \beta_2$ | $\alpha_1 = \alpha_2,\ \gamma_1 = \gamma_2\ [\beta_1,\beta_2 \neq 0 \text{ or } \pi]$ (in)equalities modulo 2π |
| $\beta_1 + \beta_2 = 2\pi$ | $\lvert\alpha_2 - \alpha_1\rvert = \pi,\ \lvert\gamma_2 - \gamma_1\rvert = \pi\ [\beta_1,\beta_2 \neq 0 \text{ or } \pi]$ (in)equalities modulo 2π |
| $\beta_1 = \beta_2 = \pi$ | $\alpha_1 - \gamma_1 = \alpha_2 - \gamma_2$ equality modulo 2π |
| $\beta_1 = \beta_2 = 0$ | $\alpha_1 + \gamma_1 = \alpha_2 + \gamma_2$ equality modulo 2π |

### 6.5.1.3 The Tait-Bryan convention *x*-y-*z*

The angle sequence $(\varphi,\theta,\psi)$ in Euler convention *x*-*y*-*z* is converted to a matrix ***M*** by forming the matrix product of the corresponding three principal rotation matrices specified in Table 6.2. The resulting matrix is given in Equation (6.14).

Conversely, given matrix ***M*** with elements $a_{ij}$, the equation may solved for the principal rotation factors $\boldsymbol{R}_z(\psi)\circ\boldsymbol{R}_y(\theta)\circ\boldsymbol{R}_x(\phi)$, and therefore solved for angles $(\varphi,\theta,\psi)$. The solution is given in Table 6.5.

$$\boldsymbol{R}_z(\psi)\circ\boldsymbol{R}_y(\theta)\circ\boldsymbol{R}_x(\phi) = \begin{pmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix} \tag{6.14}$$

**Table 6.5 — Principal factors for the Euler $x$-$y$-$z$ convention (Tait-Bryan)**

| Case | Principal factors for rotation $\boldsymbol{R}_z(\psi)\circ\boldsymbol{R}_y(\theta)\circ\boldsymbol{R}_x(\phi)$ (all angles modulo $2\pi$) | | |
|---|---|---|---|
| $a_{31}\neq\pm1$ | $\theta=\arcsin(-a_{31})$ <br> [ principal value ] <br> $-\pi/2<\theta<\pi/2$ | $\phi=\text{arctan2}(a_{32},\, a_{33})$ | $\psi=\text{arctan2}(a_{21},\, a_{11})$ |
| | $\theta=\arcsin(-a_{31})$ <br> [ $\pi-$principal value ] <br> $\pi/2<\theta<3\pi/2$ | $\phi=\text{arctan2}(-a_{32},\,-a_{33})$ | $\psi=\text{arctan2}(-a_{21},\,-a_{11})$ |
| $a_{31}=-1$ | $\theta=\pi/2$ | $\phi=\text{arctan2}(a_{12},\, a_{13})+\psi$ | any value of $\psi$ |
| $a_{31}=+1$ | $\theta=-\pi/2$ | $\phi=\text{arctan2}(-a_{12},\,-a_{13})-\psi$ | any value of $\psi$ |

In the case $a_{31}\neq\pm1$, arcsin() is multi-valued so that there are two valid solution sets depending on the quadrant selected for arcsine values[21]. The principal value solution is the commonly used one.

In the case $a_{31}=-1$, using the trigonometric identities for the difference of angles and substituting $\sin\theta=1$ and $\cos\theta=0$, the matrix reduces to:

$$\boldsymbol{R}_z(\psi)\circ\boldsymbol{R}_y\left(\frac{\pi}{2}\right)\circ\boldsymbol{R}_x(\phi)=\begin{pmatrix}0 & \sin(\phi-\psi) & \cos(\phi-\psi)\\ 0 & \cos(\phi-\psi) & -\sin(\phi-\psi)\\ -1 & 0 & 0\end{pmatrix}.$$

For this reason only the difference of the other two angles is determined as $\phi-\psi=\text{arctan2}(a_{12},\, a_{13})$. Therefore, all values are valid for $\psi$ if we set $\phi=\text{arctan2}(a_{12},\, a_{13})+\psi$. The case $a_{31}=+1$ is similar to the previous case with the sum of the angles determined by $\phi+\psi=\text{arctan2}(-a_{12},\,-a_{13})$. These two cases correspond to Equation (6.9) and are the gimbal lock cases.

As seen in the preceding tables, the three angle sequence corresponding to a given rotation or orientation operator is not unique modulo $2\pi$. Two sequences, $(\phi_1,\theta_1,\psi_1)$ and $(\phi_2,\theta_2,\psi_2)$ of $x$-y-$z$ principal factors specify the same operator if they satisfy one the criteria specified in Table 6.6.

21 Note that computer library functions such as asin() return the principal value only. The second solution for $\theta$ may be obtained by subtracting the principal value from $\pi$.

**Table 6.6 — Equivalence of *x*-*y*-*z* principal factor sequences**

| **Case** (equality modulo $2\pi$) | **Criteria for the equivalence of angle sequences $(\phi_1,\theta_1,\psi_1)$ and $(\phi_2,\theta_2,\psi_2)$ for principal factor *z*-*y*-*x* rotation or *x*-*y*-*z* orientation sequences** |
|---|---|
| $\theta_1 = \theta_2$ | $\phi_1 = \phi_2,\ \psi_1 = \psi_2\ \left[\theta_1 \neq \pm\frac{\pi}{2} \neq \theta_2\right]$ (in)equalities modulo $2\pi$ |
| $\theta_1 + \theta_2 = \pi$ | $\lvert\phi_2 - \phi_1\rvert = \pi,\ \lvert\psi_2 - \psi_1\rvert = \pi\ \left[\theta_1 \neq \pm\frac{\pi}{2} \neq \theta_2\right]$ (in)equalities modulo $2\pi$ |
| $\theta_1 = \theta_2 = \frac{\pi}{2}$ | $\phi_1 - \psi_1 = \phi_2 - \psi_2$ equality modulo $2\pi$ |
| $\theta_1 = \theta_2 = -\frac{\pi}{2}$ | $\phi_1 + \psi_1 = \phi_2 + \psi_2$ equality modulo $2\pi$ |

### 6.5.2 Matrix and axis-angle

Given a rotation matrix $\boldsymbol{R} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, the corresponding axis-angle representation $(\boldsymbol{n}, \theta)$ is determined using the procedure in 6.4.2.

An axis-angle rotation $(\boldsymbol{n},\theta)$, with $\boldsymbol{n} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^{\mathrm{T}}$, is converted to rotation matrix $\boldsymbol{R}$, using the matrix form of Rodrigues' rotation formula (Equation (6.3)).

$$\begin{aligned} \boldsymbol{R} &= \left[\boldsymbol{I}_{3\times3} + \sin(\theta)\boldsymbol{S}_{\boldsymbol{n}} + (1-\cos(\theta))\boldsymbol{S}_{\boldsymbol{n}}^2\right] \\ &= \left[\cos(\theta)\boldsymbol{I}_{3\times3} + (1-\cos(\theta))\boldsymbol{n}\otimes\boldsymbol{n} + \sin(\theta)\boldsymbol{S}_{\boldsymbol{n}}\right] \end{aligned} \tag{6.15}$$

where:

$\boldsymbol{S}_{\boldsymbol{n}} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$ is the skew-symmetric matrix associated with $\boldsymbol{n} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^{\mathrm{T}}$ and

$\boldsymbol{n}\otimes\boldsymbol{n} = \begin{pmatrix} n_1n_1 & n_1n_2 & n_1n_3 \\ n_2n_1 & n_2n_2 & n_2n_3 \\ n_3n_1 & n_3n_2 & n_3n_3 \end{pmatrix}$ is the outer-product of $\boldsymbol{n}$ with $\boldsymbol{n}$.

The equation expands to yield matrix elements:

$$\boldsymbol{R}=\begin{pmatrix}(1-\cos\theta)n_1^2+\cos\theta & (1-\cos\theta)n_1n_2-n_3\sin\theta & (1-\cos\theta)n_1n_3+n_2\sin\theta\\ (1-\cos\theta)n_2n_1+n_3\sin\theta & (1-\cos\theta)n_2^2+\cos\theta & (1-\cos\theta)n_2n_3-n_1\sin\theta\\ (1-\cos\theta)n_3n_1-n_2\sin\theta & (1-\cos\theta)n_3n_2+n_1\sin\theta & (1-\cos\theta)n_3^2+\cos\theta\end{pmatrix} \tag{6.16}$$

### 6.5.3 Axis-angle and quaternion

A rotation in axis-angle form $(\boldsymbol{n},\theta)$ corresponds to unit quaternion $\boldsymbol{q}=\left(\cos(\theta/2),\ \sin(\theta/2)\boldsymbol{n}\right)$.

A unit quaternion corresponds to axis-angle form $(\boldsymbol{n},\theta)$ computed as in Equation (6.10).

### 6.5.4 Matrix and quaternion

The matrix $\boldsymbol{M}$ corresponding to a unit quaternion $\boldsymbol{q}=(e_0,\boldsymbol{e})$, $\boldsymbol{e}=(e_1,e_2,e_3)^{\mathrm{T}}$ is

$$\boldsymbol{M}=\begin{pmatrix}1-2\left(e_2^2+e_3^2\right) & 2\left(e_1e_2-e_0e_3\right) & 2\left(e_1e_3+e_0e_2\right)\\ 2\left(e_1e_2+e_0e_3\right) & 1-2\left(e_1^2+e_3^2\right) & 2\left(e_2e_3-e_0e_1\right)\\ 2\left(e_1e_3-e_0e_2\right) & 2\left(e_2e_3+e_0e_1\right) & 1-2\left(e_1^2+e_2^2\right)\end{pmatrix} \tag{6.17}$$

The quaternion $\boldsymbol{q}$ corresponding to a rotation matrix $\boldsymbol{M}=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ is computed as follows:

$$e_0^2=\tfrac{1}{4}\left(1+\mathrm{Trace}(\boldsymbol{R})\right)=\tfrac{1}{4}\left(1+a_{11}+a_{22}+a_{33}\right)$$

if $e_0^2>0$,

$$\boldsymbol{e}=\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix}=\frac{1}{4e_0}\begin{pmatrix}a_{32}-a_{23}\\ a_{13}-a_{31}\\ a_{21}-a_{12}\end{pmatrix},$$

else $e_0=0$,

$$e_1^2=-\tfrac{1}{2}\left(a_{22}+a_{33}\right),$$

if $e_1^2>0$, $e_2=\dfrac{a_{12}}{2e_1}$, $e_3=\dfrac{a_{13}}{2e_1}$,

else $e_1=0$,

$$e_2^2=\tfrac{1}{2}\left(1-a_{33}\right),$$

if $e_2^2>0$, $e_3=\dfrac{a_{23}}{2e_2}$

else $e_2=0$, $e_3=1$.

A rotationally equivalent quaternion is $-\boldsymbol{q}$.

### 6.5.5 Euler angle conventions and quaternions

The principal rotations (see 6.4.4.1) correspond to the following quaternions:

$$\boldsymbol{R}_z(\gamma) \leftrightarrow \left(\cos(\gamma/2), \sin(\gamma/2)\boldsymbol{z}\right)$$
$$\boldsymbol{R}_y(\beta) \leftrightarrow \left(\cos(\beta/2), \sin(\beta/2)\boldsymbol{y}\right)$$
$$\boldsymbol{R}_x(\alpha) \leftrightarrow \left(\cos(\alpha/2), \sin(\alpha/2)\boldsymbol{x}\right)$$

For each Euler angle convention, multiply the corresponding quaternions in the space-fixed realization ordering. Terms in the resulting product may be simplified using the orthonormal property of the vector set $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$, and various trigonometric identities.

For the Euler angle $z$-$x$-$z$ convention, the quaternion $\boldsymbol{q}$ corresponding to $\boldsymbol{R}_z(\gamma) \circ \boldsymbol{R}_x(\beta) \circ \boldsymbol{R}_z(\alpha)$ is:

$$\boldsymbol{q} = \left(\cos(\gamma/2), \sin(\gamma/2)\boldsymbol{z}\right)\left(\cos(\beta/2), \sin(\beta/2)\boldsymbol{x}\right)\left(\cos(\alpha/2), \sin(\alpha/2)\boldsymbol{z}\right).$$

Multiplied out the expression reduces to:

$$\boldsymbol{q} = (e_0, \boldsymbol{e})$$

where:

$$e_0 = \cos\left(\frac{\gamma+\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right),$$

$$\boldsymbol{e} = \left(\cos\left(\frac{\gamma-\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right) \quad \sin\left(\frac{\gamma-\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right) \quad \sin\left(\frac{\gamma+\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\right)^{\mathrm{T}}$$

For the Euler angle $x$-$y$-$z$ convention (Tait-Bryan angles), the quaternion $\boldsymbol{q}$ corresponding to $\boldsymbol{R}_z(\psi) \circ \boldsymbol{R}_y(\theta) \circ \boldsymbol{R}_x(\phi)$ is:

$$\boldsymbol{q} = \left(\cos(\psi/2), \sin(\psi/2)\boldsymbol{z}\right)\left(\cos(\theta/2), \sin(\theta/2)\boldsymbol{y}\right)\left(\cos(\phi/2), \sin(\phi/2)\boldsymbol{x}\right).$$

Multiplied out the expression reduces to:

$$\boldsymbol{q} = (e_0, \boldsymbol{e}) = (e_0, e_1, e_2, e_3)$$

where:

$$e_0 = \cos(\psi/2)\cos(\theta/2)\cos(\phi/2) + \sin(\psi/2)\sin(\theta/2)\sin(\phi/2)$$
$$e_1 = \cos(\psi/2)\cos(\theta/2)\sin(\phi/2) - \sin(\psi/2)\sin(\theta/2)\cos(\phi/2)$$
$$e_2 = \cos(\psi/2)\sin(\theta/2)\cos(\phi/2) + \sin(\psi/2)\cos(\theta/2)\sin(\phi/2)$$
$$e_3 = \sin(\psi/2)\cos(\theta/2)\cos(\phi/2) - \cos(\psi/2)\sin(\theta/2)\sin(\phi/2)$$

To convert a unit quaternion $\boldsymbol{q} = (e_0, \boldsymbol{e}) = (e_0, e_1, e_2, e_3)$ to the Euler angle $z$–$x$–$z$ convention $\boldsymbol{R}_z(\gamma) \circ \boldsymbol{R}_x(\beta) \circ \boldsymbol{R}_z(\alpha)$, compute as follows:

if $0 < \left(e_1^2 + e_2^2\right) < 1$:

$$\alpha = \operatorname{arctan2}\left((e_1e_3 + e_0e_2), -(e_2e_3 - e_0e_1)\right)$$
$$\beta = \arccos\left(1 - 2\left(e_1^2 + e_2^2\right)\right) \qquad \text{principal value: } 0 < \beta < \pi$$
$$\gamma = \operatorname{arctan2}\left((e_1e_3 - e_0e_2), (e_2e_3 + e_0e_1)\right)$$

if $\left(e_1^2+e_2^2\right)=0$: $\beta=0$ and $\alpha+\gamma=\text{arctan2}\left(\left(e_1e_2-e_0e_3\right),\tfrac{1}{2}-\left(e_2^2+e_3^2\right)\right)$.

if $\left(e_1^2+e_2^2\right)=1$: $\beta=\pi$ and $\alpha-\gamma=\text{arctan2}\left(\left(e_1e_2-e_0e_3\right),\tfrac{1}{2}-\left(e_2^2+e_3^2\right)\right)$.

The solution in the first case is not unique, see Table 6.4. The last two cases are Euler angle gimbal lock cases.

To convert a unit quaternion $\boldsymbol{q}=\left(e_0,\boldsymbol{e}\right)=\left(e_0,e_1,e_2,e_3\right)$ to the Euler angle $x$-$y$-$z$ convention (Tait-Bryan angles) $\boldsymbol{R}_z\left(\psi\right)\circ\boldsymbol{R}_y\left(\theta\right)\circ\boldsymbol{R}_x\left(\phi\right)$, compute as follows.

If $2\left(e_1e_3-e_0e_2\right)\neq\pm1$:

$\phi=\text{arctan2}\left(\left(e_2e_3+e_0e_1\right),\tfrac{1}{2}-\left(e_1^2+e_2^2\right)\right)$

$\theta=\text{arcsin}\left(-2\left(e_1e_3-e_0e_2\right)\right)$ principal value: $-\pi/2<\theta<\pi/2$

$\psi=\text{arctan2}\left(\left(e_1e_2+e_0e_3\right),\tfrac{1}{2}-\left(e_2^2+e_3^2\right)\right)$

If $2\left(e_1e_3-e_0e_2\right)=+1$: $\theta=-\pi/2$ and $\phi+\psi=\text{arctan2}\left(\left(e_1e_2-e_0e_3\right),\left(e_1e_3+e_0e_2\right)\right)$.

If $2\left(e_1e_3-e_0e_2\right)=-1$: $\theta=\pi/2$ and $\phi-\psi=\text{arctan2}\left(\left(e_1e_2-e_0e_3\right),\left(e_1e_3+e_0e_2\right)\right)$.

The solution in the first case is not unique, see Table 6.6. The last two cases are Euler angle gimbal lock cases.



 