# 10 Operations

## 10.1 Introduction

This International Standard specifies operations on SRF coordinates and, in the case of 3D object-spaces, on SRF spatial directions, vectors and orientations. Underlying these operations are the similarity transformations relating two ORMs. Similarity transformations are treated first in 10.3. The general case of changing the representation of a position as a coordinate in one SRF to its representation as a coordinate in another SRF is specified in 10.4, followed by important special cases. The specification of a spatial direction, vector or orientation in the context of an SRF is defined, and operations for changing these representations from one SRF to their corresponding representations in another SRF are specified in 10.5.

Euclidean distance in 2D and 3D object-space is specified in 10.6. Geodesic distance and azimuth on the surface of an oblate ellipsoid (or sphere) are specified in 10.7.

## 10.2 Notation and terminology

An important category of spatial operations is changing the representation of spatial information in one SRF to the representation in a second SRF. For these change of SRF operations, the adjective "source" shall be used to refer to the first SRF, and the adjective "target" shall be used to refer to the second SRF.

The notation in Table 10.1 is used throughout this clause.

**Table 10.1 — Notation**

| Notation | Description |
|---|---|
| $\mathrm{ORM_S}$ | Source ORM |
| $\mathrm{ORM_T}$ | Target ORM |
| $\mathrm{ORM_R}$ | Reference ORM for a given spatial object |
| $\boldsymbol{H}_{\mathrm{RT}}$ | Reference transformation from the reference $\mathrm{ORM_R}$ to $\mathrm{ORM_T}$ |
| $\boldsymbol{H}_{\mathrm{SR}}$ | Reference transformation from $\mathrm{ORM_S}$ to the reference $\mathrm{ORM_R}$ |
| $\boldsymbol{H}_{\mathrm{TR}}$ | Reference transformation from $\mathrm{ORM_T}$ to the reference $\mathrm{ORM_R}$ |
| $\boldsymbol{H}_{\mathrm{ST}}$ | Similarity transformation from the embedding of $\mathrm{ORM_S}$ to $\mathrm{ORM_T}$ |
| $\boldsymbol{M}_{\mathrm{RT}}$ | Rotation matrix from the reference $\mathrm{ORM_R}$ to $\mathrm{ORM_T}$ |
| $\boldsymbol{M}_{\mathrm{SR}}$ | Rotation matrix from $\mathrm{ORM_S}$ to the reference $\mathrm{ORM_R}$ |
| $\boldsymbol{M}_{\mathrm{ST}}$ | Rotation matrix from $\mathrm{ORM_S}$ to $\mathrm{ORM_T}$ |
| $\boldsymbol{M}_{\mathrm{TR}}$ | Rotation matrix from $\mathrm{ORM_T}$ to the reference $\mathrm{ORM_R}$ |
| $\mathrm{SRF_S}$ | Source SRF based on $\mathrm{ORM_S}$ |
| $\mathrm{SRF_T}$ | Target SRF based on $\mathrm{ORM_T}$ |
| $\mathrm{SRF_L}$ | The local tangent frame SRF at a coordinate (See 10.5.2) |
| $\mathrm{CS_S}$ | CS of $\mathrm{SRF_S}$ |
| $\mathrm{CS_T}$ | CS of $\mathrm{SRF_T}$ |
| $\boldsymbol{G}_{\mathrm{S}}$ | Generating function of $\mathrm{CS_S}$ |
| $\boldsymbol{G}_{\mathrm{T}}^{-1}$ | Inverse generating function of $\mathrm{CS_T}$ |
| $\boldsymbol{c}_{\mathrm{S}}$ | Coordinate of a spatial position in $\mathrm{SRF_S}$ |

| | |
|---|---|
| $\boldsymbol{c}_{\mathrm{T}}$ | Coordinate of a spatial position in $\mathrm{SRF_T}$ |
| $\boldsymbol{n}_{\mathrm{S}}$ | Direction vector in $\mathrm{SRF_S}$ (See 10.5.2) |
| $\boldsymbol{n}_{\mathrm{T}}$ | Direction vector in $\mathrm{SRF_T}$ (See 10.5.2) |
| $\boldsymbol{R}_{\mathrm{ST}}$ | Orientation of $\mathrm{SRF_T}$ with respect to $\mathrm{SRF_S}$ in the position vector rotation convention |

## 10.3 Operations on ORMs

### 10.3.1 Introduction

The similarity transformation (see 7.3.2) $\boldsymbol{H}_{\mathrm{ST}}$ between a source/target pair $\mathrm{ORM_S}$ and $\mathrm{ORM_T}$ underlies the coordinate operations in 10.4. Given a set of $n$ ORMs there are $n(n\text{-}1)$ such source and target ORM pairs. Instead of specifying the full set of similarity transformations, this International Standard reduces the requirement to specifying the reference transformation $\boldsymbol{H}_{\mathrm{SR}}$ from each object-fixed source $\mathrm{ORM_S}$ to the reference $\mathrm{ORM_R}$ for a given object. This subclause specifies the methods of expressing a similarity transformation $\boldsymbol{H}_{\mathrm{ST}}$ in terms of the reference transformations for the source and target ORMs. The cases of ORMs for a single object are treated in 10.3.2. The more general cases in which $\mathrm{ORM_S}$ and $\mathrm{ORM_T}$ are ORMs for different objects are treated in 10.3.3.

### 10.3.2 ORMs for a single object

If $\mathrm{ORM_S}$ is an object-fixed ORM, its reference transformation $\boldsymbol{H}_{\mathrm{SR}}$ is a type of similarity transformation. Any 3D or 2D similarity transformation may be represented with the STT ROTATE_SCALE_TRANSLATE in the 3D case or STT ROTATE_SCALE_TRANSLATE_2D in the 2D case (see Tables 7.19 and 7.20). Thus using the notation of the STT formulation, $\boldsymbol{H}_{\mathrm{SR}}$ may be represented in the form given by Equation (10.1).

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{R}} = \boldsymbol{H}_{\mathrm{SR}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}}\right) \equiv \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{SR}} + s_{\mathrm{SR}} \boldsymbol{M}_{\mathrm{SR}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}} \tag{10.1}$$

NOTE The processes by which ORMs for the Earth are established are based on physical measurements. These measurements are subject to error, and therefore introduce various types of relative distortions between ORMs. Equation (10.1) is based on the assumption that positions in object-space are error free, and the equation includes no compensation for these distortions.

The reference transformation $\boldsymbol{H}_{\mathrm{TR}}$ from $\mathrm{ORM_T}$ to the reference $\mathrm{ORM_R}$ is also a similarity transformation.

An important operation is the similarity transformation $\boldsymbol{H}_{\mathrm{ST}}$ from $\mathrm{ORM_S}$ to $\mathrm{ORM_T}$, when neither the source nor the target is necessarily the reference ORM. The $\boldsymbol{H}_{\mathrm{ST}}$ transformation may be expressed as the composition of $\boldsymbol{H}_{\mathrm{SR}}$ with $\boldsymbol{H}_{\mathrm{TR}}^{-1}$ (or $\boldsymbol{H}_{\mathrm{RT}}$, which is equivalent to the inverse of $\boldsymbol{H}_{\mathrm{TR}}$) as in Equation (10.2) (see Figure 10.1):

$$\boldsymbol{H}_{\mathrm{ST}} = \boldsymbol{H}_{\mathrm{RT}} \circ \boldsymbol{H}_{\mathrm{SR}} \tag{10.2}$$

 

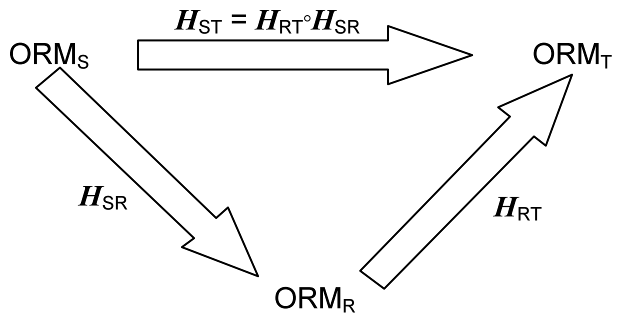


**Figure 10.1 — Composed transformations**

$\boldsymbol{H}_{\mathrm{RT}}$ is also a similarity transformation:

$$\boldsymbol{H}_{\mathrm{RT}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{R}}\right) = \frac{1}{s_{\mathrm{TR}}}\boldsymbol{M}_{\mathrm{TR}}^{-1}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{R}} - \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{ST}}\right)$$
$$= \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{RT}} + \frac{1}{s_{\mathrm{TR}}}\boldsymbol{M}_{\mathrm{TR}}^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{R}}$$

Because the matrix $\boldsymbol{M}_{\mathrm{TR}}$ is a rotation matrix, its transpose $\boldsymbol{M}_{\mathrm{TR}}^{\boldsymbol{T}}$ is also its inverse $\boldsymbol{M}_{\mathrm{TR}}^{-1}$. The inverse of $\boldsymbol{M}_{\mathrm{TR}}$ is also the matrix $\boldsymbol{M}_{\mathrm{RT}}$ corresponding to the reverse rotations of ORM$_{\mathrm{T}}$ with respect to ORM$_{\mathrm{R}}$. In particular:

$$\boldsymbol{M}_{\mathrm{RT}} = \boldsymbol{M}_{\mathrm{TR}}^{-1} = \boldsymbol{M}_{\mathrm{TR}}^{\boldsymbol{T}}$$

and

$$\boldsymbol{H}_{\mathrm{RT}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{R}}\right) = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{RT}} + \frac{1}{s_{\mathrm{TR}}}\boldsymbol{M}_{\mathrm{RT}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{R}}.$$

The composite operation $\boldsymbol{H}_{\mathrm{ST}} = \boldsymbol{H}_{\mathrm{RT}} \circ \boldsymbol{H}_{\mathrm{SR}}$ reduces to:

$$\boldsymbol{H}_{\mathrm{ST}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}}\right) = \boldsymbol{H}_{\mathrm{RT}} \circ \boldsymbol{H}_{\mathrm{SR}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}}\right) = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{ST}} + \frac{s_{\mathrm{SR}}}{s_{\mathrm{TR}}}\boldsymbol{M}_{\mathrm{ST}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}} \tag{10.3}$$

where:

$$\boldsymbol{M}_{\mathrm{ST}} = \boldsymbol{M}_{\mathrm{RT}} \circ \boldsymbol{M}_{\mathrm{SR}}, \text{ and}$$
$$\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{ST}} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{RT}} + \frac{1}{s_{\mathrm{TR}}}\boldsymbol{M}_{\mathrm{RT}}\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{SR}}.$$

If the rotations $\boldsymbol{M}_{\mathrm{SR}}$ and $\boldsymbol{M}_{\mathrm{TR}}$ are equal, then $\boldsymbol{M}_{\mathrm{ST}}$ is the identity matrix, and if $s_{\mathrm{SR}} = s_{\mathrm{TR}}$, $\boldsymbol{H}_{\mathrm{ST}}$ simplifies to a translation of the origin:

$$\boldsymbol{H}_{\mathrm{ST}}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}}\right) = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{\mathrm{ST}} + \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{S}}.$$

 

Equation (10.2) and Figure 10.1 also apply to the 2D case.

If the source ORM$_S$ is a time-dependent ORM for a spatial object, ORM$_S(t)$ shall denote the ORM$_S$ at time $t$, and $\boldsymbol{H}_{SR}(t)$ shall denote the similarity transformation from ORM$_S(t)$ to the object-fixed reference ORM$_R$. If the similarity transformation $\boldsymbol{H}_{SR}(t)$ can be determined, it is a time-dependent affine transformation. For a fixed value of time $t_0$, Equation (10.2) and Figure 10.1 generalize to $\boldsymbol{H}_{ST}(t_0) = \boldsymbol{H}_{RT} \circ \boldsymbol{H}_{SR}(t_0)$. The generalization to a time-dependent target ORM$_T(t)$ is $\boldsymbol{H}_{ST}(t_0) = \boldsymbol{H}_{RT}(t_0) \circ \boldsymbol{H}_{SR}$. The generalization when both ORMs are time-dependent at time $t_0$ is $\boldsymbol{H}_{ST}(t_0) = \boldsymbol{H}_{RT}(t_0) \circ \boldsymbol{H}_{SR}(t_0)$.

EXAMPLE ORM$_S(t)$ is the ORM EARTH_INERTIAL_J2000r0 at time $t$. ORM$_R$ is the Earth reference ORM WGS_1984. Because ORM$_S(t)$ and ORM$_R$ share the same embedding origin, the $\boldsymbol{H}_{SR}(t)$ transformation is a (rotation) matrix multiplication operation (without translation). The matrix coefficients for selected values of $t$ account for polar motion, Earth rotation, nutation, and precession. Predicted values for these coefficients are computed and updated weekly by the International Earth Rotation and Reference Systems Service (IERS) [IERS36]. See 7.5 for other examples of dynamic ORM reference transformations.

### 10.3.3 Relating ORMs for different objects

If a spatial object ***S*** exists in the space of another spatial object ***T***, and if ORM$_R$ is the reference ORM for object ***T***, and if the two objects are fixed with respect to each other, then $\boldsymbol{H}_{SR}$ shall denote a similarity transformation from ORM$_S$ to ORM$_R$. $\boldsymbol{H}_{SR}$ is an affine transformation. If ORM$_T$ is an object-fixed ORM for the object ***T,*** then $\boldsymbol{H}_{ST}$ is given by Equation (10.2). The time dependent generalizations of Equation (10.2), defined in 10.3.2, are also applicable to this case.

EXAMPLE ORM$_S$ is an ORM for the space shuttle (as a spatial object). ORM$_R$ is the Earth reference ORM WGS_1984. When in orbit at time $t$, $\boldsymbol{H}_{SR}(t)$ transforms positions with respect to ORM$_S$ to positions with respect to ORM WGS_1984.

If a spatial object ***S*** does not exist in the space of another spatial object ***T***, a similarity transformation between their ORMs is not intrinsically determined. However, if an invertible affine transformation ($\boldsymbol{H}_{SR}$) between ORM$_S$ and the reference ORM for object ***T*** is provided, then, given an object-fixed ORM for object ***T,*** ORM$_T$, Equation (10.2) may be used to define an invertible affine transformation $\boldsymbol{H}_{ST}$, from ORM$_S$ to ORM$_T$. An important instance of this case occurs when ***S*** is an abstract object and ***T*** is a physical object (see 10.4.6).

## 10.4 Operations to change spatial coordinates between SRFs

### 10.4.1 Introduction

Given a coordinate $c_S$ in a source SRF, SRF$_S$, the change of SRF operation[25] computes the corresponding coordinate $c_T$ in a given target SRF, SRF$_T$. The general case of this operation is presented in formulations in 10.4.2 for time-independent (static) and time-dependent (dynamic) ORM relationships. The specific SRF coordinate-systems CS$_S$ and CS$_T$ impose restrictions on the applicability of the formulation because of CS domain/range constraints (see below).

The formulations depend on the existence of a (static or dynamic) embedding transformation $\boldsymbol{H}_{ST}$ from ORM$_S$ to ORM$_T$. If ORM$_S$ and ORM$_T$ have the same object space, $\boldsymbol{H}_{ST}$ is formulated in 10.3.2 in terms of ORM specification elements. In the case of different object spaces, $\boldsymbol{H}_{ST}$ must be explicitly provided (see 10.3.2).

Special cases allow for simplifications that result in computational short cuts to the general change of SRF formulation. The case of matched normal embeddings (which includes the case ORM$_S$ = ORM$_T$) is treated in 10.4.3. Further specializations arise from combinations of specific coordinate-systems. Subclause 10.4.4 treats combinations of celestiodetic with a map projection.

---

[25] ISO 19111 defines this case as a coordinate operation.

 

The case for which $CS_S = CS_T$ and $ORM_S$ and $ORM_T$ differ[26] does not generally produce a computational simplification. However, when both the source and target SRFs are based on the CS LOCOCENTRIC_EUCLIDEAN_3D, a simplification is produced and is presented in 10.4.5. This case is important for operations on directions, vectors, and orientations (see 10.5).

An important special case of unrelated object spaces occurs when the source object space is an abstract 3D object space. This special case is treated in 10.4.6.

### 10.4.2 General case

$SRF_S$ and $SRF_T$ are two object-fixed SRFs for a spatial object and $\boldsymbol{p}$ is a point in object-space that is in the coordinate system domains of both SRFs. $\boldsymbol{c}_S$ denotes the coordinate of $\boldsymbol{p}$ in $SRF_S$, and $\boldsymbol{c}_T$ denotes the coordinate of $\boldsymbol{p}$ in $SRF_T$. The determination of $\boldsymbol{c}_T$ from $\boldsymbol{c}_S$ is an operation involving the SRF pair ($SRF_S$, $SRF_T$). The most general form of the operation is:

$$\boldsymbol{c}_T = \boldsymbol{G}_T^{-1} \circ \boldsymbol{H}_{ST} \circ \boldsymbol{G}_S(\boldsymbol{c}_S) \tag{10.4}$$

See Figure 10.2. CS generating and inverse generating functions are specified in Clause 5.

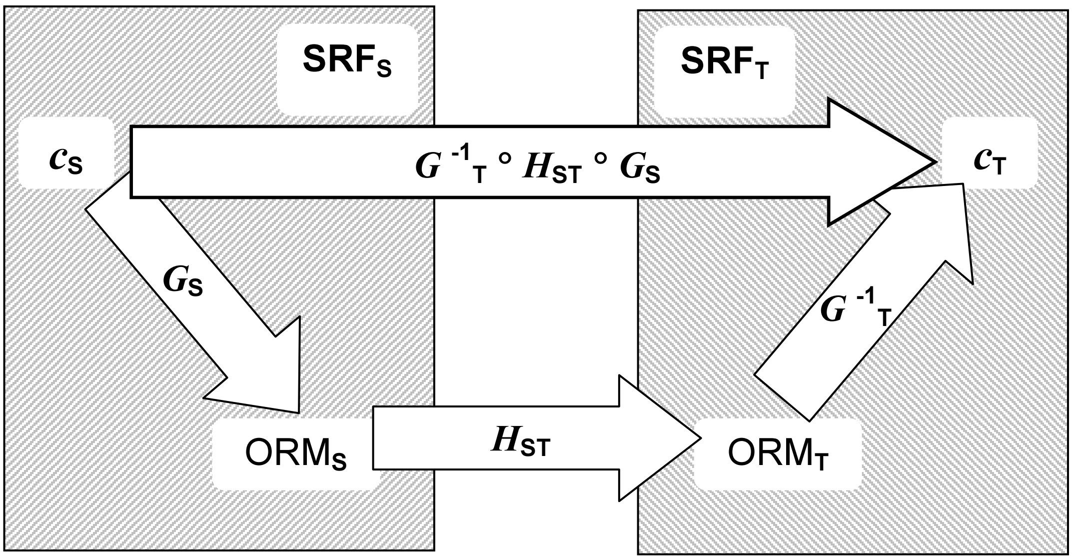


**Figure 10.2 — Change of SRF operation – applied to coordinates**

When $\boldsymbol{H}_{ST}$ is approximated with the Bursa-Wolf equation (see STT PV_7_PARAMETER Note 2), Equation (10.4) is known as the *Helmert transformation*.

[26] ISO 19111 defines this case as a coordinate transformation.

If $\mathrm{SRF_S}$ and $\mathrm{SRF_T}$ are two celestiodetic SRFs with different ORMs for the same spatial object, Equation (10.4) transforms the coordinate $\boldsymbol{c}_\mathrm{S} = (\lambda_\mathrm{S}, \varphi_\mathrm{S}, h_\mathrm{S})$ with respect to one oblate ellipsoid to $\boldsymbol{c}_\mathrm{T} = (\lambda_\mathrm{T}, \varphi_\mathrm{T}, h_\mathrm{T})$ with respect to the other oblate ellipsoid. A transformation between two celestiodetic SRFs for the spatial object Earth is known as a *horizontal datum shift*.

NOTE A number of numerical approximations developed to implement horizontal datum shift have been published. Under the assumption of zero rotations and no scale differences, a widely used approximation[27] to directly transform $\boldsymbol{c}_\mathrm{S} = (\lambda_\mathrm{S}, \varphi_\mathrm{S}, h_\mathrm{S})$ to $\boldsymbol{c}_\mathrm{T} = (\lambda_\mathrm{T}, \varphi_\mathrm{T}, h_\mathrm{T})$ is the *standard Molodensky transformation* formula (see [83502T]).

In the time-dependent case, Equation (10.4) may be generalized to:

$$\boldsymbol{c}_\mathrm{T}(t) = \boldsymbol{G}_\mathrm{T}^{-1} \circ \boldsymbol{H}_\mathrm{ST}(t) \circ \boldsymbol{G}_\mathrm{S}(\boldsymbol{c}_\mathrm{S})$$

Equation (10.4) is only defined for a value of $\boldsymbol{c}_\mathrm{S}$ in the $\mathrm{CS_S}$ domain if its corresponding position belongs to the $\mathrm{CS_T}$ range (the range of a generating function is the domain of its inverse generating function). If $R_\mathrm{S}$ is the range of the generating function $\boldsymbol{G}_\mathrm{S}$ and $R_\mathrm{T}$ is the range of the generating function $\boldsymbol{G}_\mathrm{T}$, Equation (10.4) is only defined for $\boldsymbol{c}_\mathrm{S}$ in the set:

$$\boldsymbol{G}_\mathrm{S}^{-1}\left(R_\mathrm{S} \cap \boldsymbol{H}_\mathrm{ST}^{-1}(R_\mathrm{T})\right) \equiv \left\{\boldsymbol{c}_\mathrm{S} \text{ in } D_\mathrm{S} \mid \boldsymbol{H}_\mathrm{ST}\left(\boldsymbol{G}_\mathrm{S}(\boldsymbol{c}_\mathrm{S})\right) \text{ in } R_\mathrm{T}\right\} \tag{10.5}$$

If $\boldsymbol{c}_\mathrm{S}$ does not belong to this set, it is invalid for the operation in Equation (10.4).

EXAMPLE $\mathrm{SRF_S}$ is SRF GEOCENTRIC_WGS_1984 and $\mathrm{SRF_T}$ is an instance of SRFT MERCATOR, with ORM WGS_1984. Equation (10.4) is not defined for any $\boldsymbol{c}_\mathrm{S}$ that is on the $z$-axis of $\mathrm{SRF_S}$, because the $z$-axis is not contained in the set in Equation (10.5).

$\mathrm{SRF_T}$ may optionally specify an SRF region $V_\mathrm{T}$, and may optionally also specify an extended SRF region $E_\mathrm{T}$ (see 8.3.2.4). If $D_\mathrm{T}$ is the domain of the generating function $\boldsymbol{G}_\mathrm{T}$, then $V_\mathrm{T} \subseteq E_\mathrm{T} \subseteq D_\mathrm{T}$. If $\boldsymbol{c}_\mathrm{T}$ is computed using Equation (10.4), $\boldsymbol{c}_\mathrm{T}$ is either within the SRF region ($\boldsymbol{c}_\mathrm{T}$ is in $V_\mathrm{T}$), or $\boldsymbol{c}_\mathrm{T}$ is within the extended SRF region but not within the SRF region ($\boldsymbol{c}_\mathrm{T}$ is in $E_\mathrm{T} \setminus V_\mathrm{T}$), or $\boldsymbol{c}_\mathrm{T}$ is within the CS domain but not within the extended SRF region ($\boldsymbol{c}_\mathrm{T}$ is in $D_\mathrm{T} \setminus E_\mathrm{T}$).

In applications that functionally conform to an SRM profile, the domain of an SRF operation is restricted to the accuracy domain of the SRF as specified by that profile (see Clause 12).

### 10.4.3 The matched normal embeddings case

If both source and target ORMs are the same, or, more generally, if the reference transformations of $\mathrm{ORM_S}$ and $\mathrm{ORM_T}$ are equivalent (*i.e.*, matched normal embeddings), $\boldsymbol{H}_\mathrm{ST}$ is the identity transformation. Consequently, Equation (10.4) simplifies to:

$$\boldsymbol{c}_\mathrm{T} = \boldsymbol{G}_\mathrm{T}^{-1} \circ \boldsymbol{G}_\mathrm{S}(\boldsymbol{c}_\mathrm{S}). \tag{10.6}$$

EXAMPLE 1 If $\mathrm{SRF_S}$ is a celestiodetic SRF (see 8.4) and $\mathrm{SRF_T}$ is the celestiocentric SRF for the same ORM ($\mathrm{ORM_S} = \mathrm{ORM_T}$), then since the CS of the celestiocentric SRF is Euclidean_3D for which the $\boldsymbol{G}_\mathrm{T}^{-1}$ is the identity, Equation (10.6) reduces to the geodetic generating function: $\boldsymbol{c}_\mathrm{T} = \boldsymbol{G}_\mathrm{S}(\boldsymbol{c}_\mathrm{S})$.

---

[27] Historically it was thought that these approximations would require less computation than direct conversion. The perceived computational advantage may have been overcome by technology advances. New efficient algorithms for converting celestiocentric coordinates to celestiodetic coordinates have been developed that result in appreciably faster transformations without the attendant loss of accuracy.

 

If $\mathrm{SRF_T}$ is a 3D SRF that has ellipsoidal height designated as the vertical coordinate-component of the SRF (see 8.4), and $\mathrm{SRF_S}$ is the induced zero height surface SRF, the *promotion operation* converts a surface coordinate $\boldsymbol{c}_\mathrm{S}$ in $\mathrm{SRF_S}$ to a 3D coordinate in $\mathrm{SRF_T}$ by setting the 1st and 2nd coordinate-components of $\boldsymbol{c}_\mathrm{T}$ to the 1st and 2nd coordinate-components of $\boldsymbol{c}_\mathrm{S}$ and setting the 3rd coordinate-component, ellipsoidal height, to 0. Coordinate promotion is a special case of Equation (10.6). Applicable SRFs include those based on SRFT CELESTIODETIC, PLANETODETIC, and all map projection SRFTs.

EXAMPLE 2 If $\mathrm{SRF_S}$ is an induced surface celestiodetic SRF (see 8.4) and $\mathrm{SRF_T}$ is the 3D celestiodetic SRF for the same ORM ($\mathrm{ORM_S} = \mathrm{ORM_T}$), Equation (10.6) promotes $\boldsymbol{c}_\mathrm{S} = (\lambda, \varphi)$ from a coordinate of CS type surface to $\boldsymbol{c}_\mathrm{T} = (\lambda, \varphi, 0)$ a coordinate of CS type 3D.

If $\mathrm{SRF_S}$ is a 3D SRF that has ellipsoidal height designated as the vertical coordinate-component of the SRF (see 8.4), and $\mathrm{SRF_T}$ is the induced zero height surface SRF, the *truncation operation* converts a 3D coordinate $\boldsymbol{c}_\mathrm{S}$ in $\mathrm{SRF_S}$ to a surface coordinate $\boldsymbol{c}_\mathrm{T}$, by setting the 1st and 2nd coordinate-components of $\boldsymbol{c}_\mathrm{T}$ to the 1st and 2nd coordinate-components of $\boldsymbol{c}_\mathrm{S}$. The point in object-space corresponding to $\boldsymbol{c}_\mathrm{S}$ and the point in object-space corresponding to $\boldsymbol{c}_\mathrm{T}$ are not the same point unless $h = 0$. Truncation, therefore, does not generally preserve location.

EXAMPLE 3 If $\mathrm{SRF_S}$ is a celestiodetic 3D SRF, the (induced) surface $\mathrm{SRF_T}$ is the surface celestiodetic SRF for the same ORM. The truncation operation associates $\boldsymbol{c}_\mathrm{T} = (\lambda, \varphi)$ to $\boldsymbol{c}_\mathrm{S} = (\lambda, \varphi, h)$.

### 10.4.4 Matched normal embeddings and map projection SRFs

The CS generating function $\boldsymbol{G}_\mathrm{MP}$ for a an augmented map projection SRF is implicitly defined (see 5.8.6) by the composition of the generating function for the geodetic 3D CS generating function $\boldsymbol{G}_\mathrm{GD}$ with the inverse mapping equation $\boldsymbol{Q} \equiv (Q_1, Q_2, h)$ as:

$$\boldsymbol{G}_\mathrm{MP} = \boldsymbol{G}_\mathrm{GD} \circ \boldsymbol{Q}.$$

If $\mathrm{SRF_S}$ and $\mathrm{SRF_T}$ are map projection SRFs for the same object, and the corresponding reference transformations are equivalent, then Equation (10.6) becomes:

$$\begin{aligned}\boldsymbol{c}_\mathrm{T} &= \left(\boldsymbol{G}_{GD,\mathrm{T}} \circ \boldsymbol{Q}_\mathrm{T}\right)^{-1} \circ \left(\boldsymbol{G}_{GD,\mathrm{S}} \circ \boldsymbol{Q}_\mathrm{S}\right)(\boldsymbol{c}_\mathrm{S}) \\ &= \boldsymbol{P}_\mathrm{T} \circ \boldsymbol{G}^{-1}_{GD,\mathrm{T}} \circ \boldsymbol{G}_{GD,\mathrm{S}} \circ \boldsymbol{Q}_\mathrm{S}(\boldsymbol{c}_\mathrm{S})\end{aligned} \qquad (10.7)$$

where:

$\boldsymbol{Q}_\mathrm{S}$ : inverse mapping equations for $\mathrm{SRF_S}$,

$\boldsymbol{G}_{GD,\mathrm{S}}$ : generating function for the geodetic 3D CS for $\mathrm{SRF_S}$,

$\boldsymbol{Q}_\mathrm{T}$ : inverse mapping equations for $\mathrm{SRF_T}$,

$\boldsymbol{P}_\mathrm{T}$ : mapping equations for $\mathrm{SRF_T}$, and

$\boldsymbol{G}_{GD,\mathrm{T}}$ : generating function for the geodetic 3D) CS for $\mathrm{SRF_T}$.

Furthermore, if $\mathrm{ORM_S} = \mathrm{ORM_T}$, then $\boldsymbol{G}_{GD,\mathrm{S}} = \boldsymbol{G}_{GD,\mathrm{T}}$ and Equation (10.7) simplifies to:

$$\boldsymbol{c}_\mathrm{T} = \boldsymbol{P}_\mathrm{T} \circ \boldsymbol{Q}_\mathrm{S}(\boldsymbol{c}_\mathrm{S}). \qquad (10.8)$$

If $\mathrm{SRF_T}$ is a celestiodetic SRF and $\mathrm{ORM_T} = \mathrm{ORM_S}$, Equation (10.6) simplifies to:

$$\boldsymbol{c}_\mathrm{T} = \boldsymbol{Q}_\mathrm{S}(\boldsymbol{c}_\mathrm{S}).$$

Similarly, if $\mathrm{SRF_S}$ is a celestiodetic SRF and $\mathrm{ORM_T} = \mathrm{ORM_S}$, Equation (10.6) simplifies to:

$$\boldsymbol{c}_\mathrm{T} = \boldsymbol{P}_\mathrm{T}(\boldsymbol{c}_\mathrm{S}).$$

### 10.4.5 Linear orthonormal 3D SRFs

The special case of source and target SRFs based on the CS LOCOCENTRIC_EUCLIDEAN_3D is important for the treatment of directions, vectors, and orientations (see 10.5). Every linear orthonormal CS may be viewed as an instance of a CS LOCOCENTRIC_EUCLIDEAN_3D. If $\text{SRF}_\text{S}$ and $\text{SRF}_\text{T}$ are two such SRFs (see Table 8.11), and $\boldsymbol{F}_{\text{LE3D}}$ is the CS LOCOCENTRIC_EUCLIDEAN_3D (see Table 5.9) generating function, $\boldsymbol{F}_{\text{LE3D}}$ may be expressed in terms of the CS binding parameter vectors $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ in the form of the affine transformation:

$$\begin{aligned}
\boldsymbol{F}_{\text{LE3D}}(\boldsymbol{c}) &= \boldsymbol{F}_{\text{LE3D}}\begin{pmatrix} u \\ v \\ w \end{pmatrix} \\
&= \boldsymbol{q} + u\boldsymbol{r} + v\boldsymbol{s} + w\boldsymbol{t} \\
&= \boldsymbol{q} + u\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} + v\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + w\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \\
&= \boldsymbol{q} + \boldsymbol{R}\,\boldsymbol{c}
\end{aligned}$$

where:

$$\boldsymbol{t} = (\boldsymbol{r} \times \boldsymbol{s}), \text{ and}$$

$$\boldsymbol{R} = \begin{pmatrix} r_1 & s_1 & t_1 \\ r_2 & s_2 & t_2 \\ r_3 & s_3 & t_3 \end{pmatrix}.$$

The inverse generating function is expressed as:

$$\boldsymbol{F}_{\text{LE3D}}^{-1}\left(\begin{pmatrix} u \\ v \\ w \end{pmatrix}\right) = \boldsymbol{R}^{\text{T}}\left(\begin{pmatrix} u \\ v \\ w \end{pmatrix} - \boldsymbol{q}\right)$$

where: $\boldsymbol{R}^{\text{T}}$ is the transpose of $\boldsymbol{R}$.

If vectors $\boldsymbol{q}_\text{S}$, $\boldsymbol{r}_\text{S}$, $\boldsymbol{s}_\text{S}$, $\boldsymbol{q}_\text{T}$, $\boldsymbol{r}_\text{T}$, and $\boldsymbol{s}_\text{T}$ are the CS binding parameters (see Table 8.11) for $\text{SRF}_\text{S}$ and $\text{SRF}_\text{T}$ respectively, then substituting the expression in Equation (10.3) for $\boldsymbol{H}_{\text{ST}}$, Equation (10.4) specializes to:

$$\begin{aligned}
\boldsymbol{c}_\text{T} &= \boldsymbol{F}_{\text{LE3D, T}}^{-1} \circ \boldsymbol{H}_{\text{ST}} \circ \boldsymbol{F}_{\text{LE3D, S}}(\boldsymbol{c}_\text{S}) \\
&= \boldsymbol{R}_\text{T}^\text{T}\left(\boldsymbol{H}_{\text{ST}}(\boldsymbol{q}_\text{S} + \boldsymbol{R}_\text{S}\boldsymbol{c}_\text{S}) - \boldsymbol{q}_\text{T}\right) \\
&= \boldsymbol{R}_\text{T}^\text{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\text{ST}} + \left(\frac{s_{\text{SR}}}{s_{\text{TR}}}\boldsymbol{M}_{\text{ST}}\boldsymbol{q}_\text{S}\right) - \boldsymbol{q}_\text{T}\right) + \frac{s_{\text{SR}}}{s_{\text{TR}}}\boldsymbol{R}_\text{T}^\text{T} \circ \boldsymbol{M}_{\text{ST}} \circ \boldsymbol{R}_\text{S}\boldsymbol{c}_\text{S}.
\end{aligned} \tag{10.9}$$

In the case that the corresponding reference transformations of $\text{ORM}_\text{S}$ and $\text{ORM}_\text{T}$ are equivalent, Equation (10.6) specializes to Equation (10.10):

$$\begin{aligned}
\boldsymbol{c}_\text{T} &= \boldsymbol{F}_{\text{LE3D, T}}^{-1} \circ \boldsymbol{F}_{\text{LE3D, S}}(\boldsymbol{c}_\text{S}) \\
&= \boldsymbol{R}_\text{T}^\text{T}(\boldsymbol{q}_\text{S} - \boldsymbol{q}_\text{T}) + \boldsymbol{R}_\text{T}^\text{T} \circ \boldsymbol{R}_\text{S}\boldsymbol{c}_\text{S}.
\end{aligned} \tag{10.10}$$

### 10.4.6 Instantiating abstract space linear SRFs

Engineering designs and abstract models are often intended for realization in the physical world.

EXAMPLE A building plan is designed in the source SRF$_S$, an abstract space LOCAL_SPACE_RECTANGULAR_3D SRF. A terrestrial site survey establishes the origin of the target SRF$_T$, a LOCAL_TANGENT_SPACE_EUCLIDEAN SRF. Source coordinates are related to target coordinates by: $(x_T, y_T, z_T) = (1+\Delta s)(x_S, y_S, z_S)$ where $(1+\Delta s)$ is a scale factor.

More generally, models are scaled, rotated, or otherwise transformed by an invertible matrix 3x3 $\boldsymbol{W}$ before a source coordinate is associated to a target coordinate. In many application domains, this similarity transformation is in the form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_T = \begin{pmatrix} x_\Delta \\ y_\Delta \\ z_\Delta \end{pmatrix} + k\boldsymbol{W}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_S$$

where $k = (1+\Delta s)$ is the scale factor, $(x_\Delta \quad y_\Delta \quad z_\Delta)$ is the translation displacement vector, and $\boldsymbol{W}$ is a rotation matrix. In the computer graphics application domain this transformation is often represented in matrix 4x4 form:

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}_T = \begin{pmatrix} a_{11} & a_{12} & a_{13} & x_\Delta \\ a_{21} & a_{22} & a_{23} & y_\Delta \\ a_{31} & a_{32} & a_{33} & z_\Delta \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}_S, \text{ where } \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = k\boldsymbol{W}.$$

This transformation between source and target coordinates may be viewed as an SRF coordinate operation from $\boldsymbol{c}_S$ in SRF$_S$, an abstract space LOCAL_SPACE_RECTANGULAR_3D SRF, to a coordinate $\boldsymbol{c}_T$ in SRF$_T$, a physical world LOCOCENTRIC_EUCLIDEAN_3D SRF.

In the notation of 10.4.5:

$$\boldsymbol{G}_S(\boldsymbol{c}_S) = \boldsymbol{R}_S\boldsymbol{c}_S, \text{ and}$$
$$\boldsymbol{G}_T(\boldsymbol{c}_T) = \boldsymbol{q}_T + \boldsymbol{R}_T\boldsymbol{c}_T.$$

Define an invertible affine transformation $\boldsymbol{H}_{ST}$ as $\boldsymbol{H}_{ST}(\boldsymbol{v}) = \boldsymbol{q}_T + \boldsymbol{R}_T \circ \boldsymbol{W}\boldsymbol{v}$ (see 10.3.3). Substitute this $\boldsymbol{H}_{ST}$ in Equation (10.4) and simplify:

$$\begin{aligned} \boldsymbol{c}_T &= \boldsymbol{G}_T^{-1} \circ \boldsymbol{H}_{ST} \circ \boldsymbol{G}_S(\boldsymbol{c}_S) \\ &= \boldsymbol{R}_T^T\left(\boldsymbol{H}_{ST}(\boldsymbol{R}_S\boldsymbol{c}_S) - \boldsymbol{q}_T\right) \\ &= \boldsymbol{R}_T^T\left(\boldsymbol{q}_T + \boldsymbol{R}_T \circ \boldsymbol{W} \circ \boldsymbol{R}_S\boldsymbol{c}_S - \boldsymbol{q}_T\right) \\ &= \boldsymbol{W} \circ \boldsymbol{R}_S\boldsymbol{c}_S \end{aligned} \tag{10.11}$$

This illustrates that the transformation $\boldsymbol{c}_T = \boldsymbol{W} \circ \boldsymbol{R}_S\boldsymbol{c}_S$ may be viewed as a change of SRF operation.

NOTE Equation (10.11) illustrates that digital graphic composite pattern modelling techniques such as SceneGraph trees that use scale and rotation matrices $\boldsymbol{W}$ together with translation operations at each tree node are special cases of Equation (10.4). See also 10.5.5 Example 2.

## 10.5 Operations on directions, vectors, and orientations

### 10.5.1 Introduction

Specification of 3D directions, vectors, or orientations associated with a 3D SRF requires an underlying 3D vector space. An SRF is either linear or curvilinear. In the linear cases, the structure of the coordinate-space provides such a 3D vector space. In particular, all lines through distinct points in a given direction $\boldsymbol{n}$ are parallel in both coordinate- and object-space. This shows that a linear SRF supports the translation invariance of directions and vectors. A linear SRF will not preserve angular relationships between directions unless the associated abstract coordinate system (CS) is also orthonormal. In the orthonormal case, angles and distances are preserved.

In the case of a curvilinear 3D SRF, the structure of the coordinate-space does not provide an underlying 3D vector space. To support curvilinear 3D SRFs, a method of associating a 3D vector space with any given reference point in the SRF shall be used. This 3D vector space is termed the local tangent frame SRF at the reference point. This association of a local tangent frame with a reference point is applied uniformly to both curvilinear and linear SRFs.

The coordinate-space of an augmented map projection SRF (a map projection augmented with ellipsoidal height as a third dimension) appears to inherit the vector-space structure of $\mathbf{R}^3$, however, the vector properties of the (easting, northing, height)-coordinates do not carry over to object-space. This is illustrated in part by the "up pointing" vector $\boldsymbol{n} = (0, 0, 1)$ that points in different spatial directions in object-space depending on the map coordinate location at which $\boldsymbol{n}$ is placed.

In Figure 10.3, distinct position points $\boldsymbol{p}$ and $\boldsymbol{q}$ on the ellipsoid surface are projected to augmented map coordinates $(s, t, 0)$ and $(u, v, 0)$. Starting at these map coordinates, the coordinates one unit away in the "up direction" are $(s, t, 1)$ and $(u, v, 1)$, respectively. In an augmented map projection, these coordinates correspond to the position-space points $\boldsymbol{p}'$ and $\boldsymbol{q}'$. The direction from $\boldsymbol{p}$ to $\boldsymbol{p}'$ is not the same as the direction from $\boldsymbol{q}$ to $\boldsymbol{q}'$. This shows that, in object-space, the "up direction" is relative to a reference point.

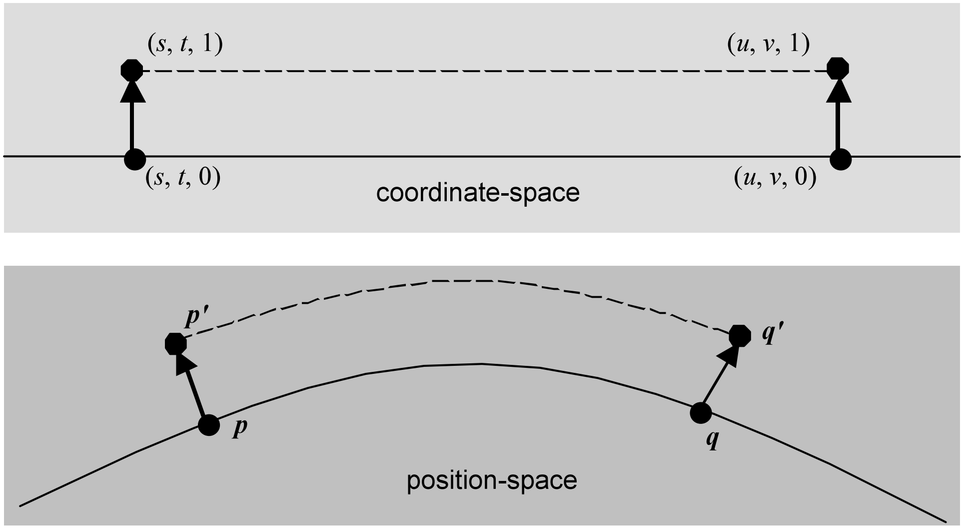


**Figure 10.3 — Coordinate-space and position-space directions compared**

A local tangent frame SRF, associated with a given reference point, shall be used to specify directions relative to that reference point. Such an SRF is defined by having its origin at the reference point and its axes given by the normalized vectors tangent to the coordinate curves passing through the reference point, as illustrated in Figure 10.4. All linear and curvilinear CSs in this International Standard are orthogonal CSs, thus the local tangent frame is an orthonormal linear SRF.

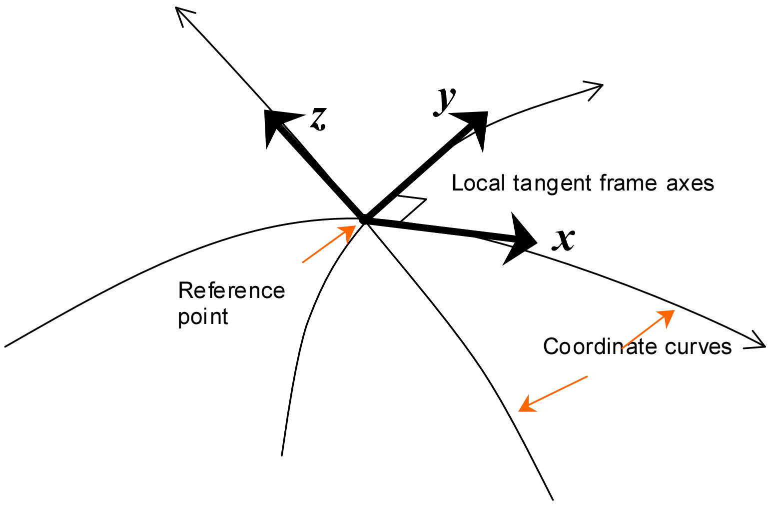


**Figure 10.4 — Local tangent frame axes**

Figure 10.5 shows two local tangent frames at points $\boldsymbol{p}$ and $\boldsymbol{q}$. The local "up" directions may be specified as a direction in either local tangent frame. Since directions are translation invariant in linear SRFs, conceptually the two local tangent frames may be translated to a common origin, as in Figure 10.6.

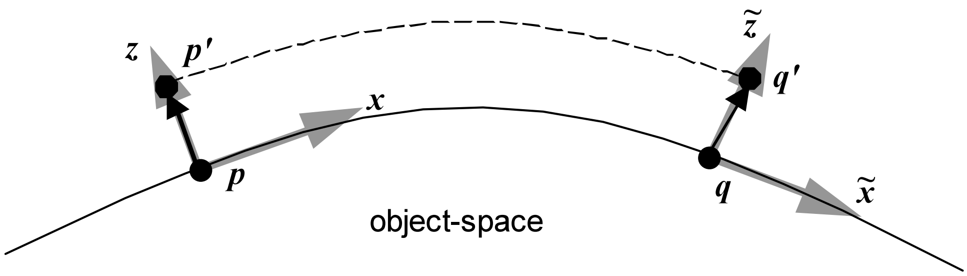


**Figure 10.5 — Local tangent frame axes at reference points *p* and *q***

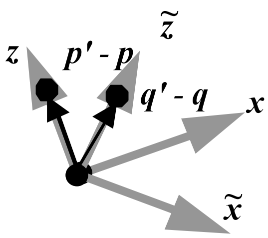


**Figure 10.6 — Direction vectors in the two local tangent frames using a common origin**

To support the inter-conversion of directions, vector quantities[28], and orientations between two SRFs, this International Standard uses the notions of reference point and local tangent frame. Since there is neither an intrinsic SRF nor an intrinsic reference point in object-space, it is necessary to specify the reference point in order to be able to inter-convert the representation of directions, vectors, or orientations between two SRFs. This method of associating reference points and local tangent frames reduces the general problem of inter-converting between two SRFs to that of inter-converting between two orthonormal linear spaces.

### 10.5.2 Specification of local tangent frame SRF

In this International Standard, a direction in a 3D[29] $\mathrm{SRF_S}$ is expressed as a combination of a unit vector and a reference coordinate. The unit vector is in a 3D linear orthonormal SRF, termed the local tangent frame at the reference coordinate, and is denoted by $\mathrm{SRF_L}$. $\mathrm{SRF_L}$ is uniquely defined for each reference coordinate using the unit vectors tangent to the coordinate-component curves at the reference coordinate.

The *local tangent frame* $\mathit{SRF}_\mathrm{L}$ at a reference coordinate $\boldsymbol{c} = (u_0, v_0, w_0)$ in the interior of the domain of $\mathrm{SRF_S}$ is specified by the SRFT LOCOCENTRIC_EUCLIDEAN_3D with ORM = $\mathrm{ORM_S}$ and parameter values:

$$\begin{aligned} \boldsymbol{q} &= \boldsymbol{G}(u_0, v_0, w_0), \\ \boldsymbol{r} &= \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}, \text{ and} \\ \boldsymbol{s} &= \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} \end{aligned} \qquad (10.12)$$

where:

$$\boldsymbol{v}_1 = \left(\frac{\mathrm{d}\boldsymbol{C}_1}{\mathrm{d}u}\right)_{u=u_0},$$

$$\boldsymbol{v}_2 = \left(\frac{\mathrm{d}\boldsymbol{C}_2}{\mathrm{d}v}\right)_{v=v_0},$$

$\boldsymbol{C}_1$ is the 1st coordinate-component curve at $(u_0, v_0, w_0)$, and

$\boldsymbol{C}_2$ is the 2nd coordinate-component curve at $(u_0, v_0, w_0)$.

The vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ are termed the *local tangent vectors at* $\boldsymbol{c}$. Coordinate-component curves are defined in 5.5.3.

NOTE The tangent vector to the 3rd coordinate-curve at $(u_0, v_0, w_0)$ points in the same direction as the vector $\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s}$ because of the coordinate-component ordering restriction specified in 5.6.4.

When $\mathrm{SRF_S}$ is a linear SRF, $\mathrm{SRF_L}$ at reference coordinate (0, 0, 0) coincides with $\mathrm{SRF_S}$. In addition, the unit vector that represents the direction is independent of the reference coordinate used. Linear SRFs include those based on SRFTs CELESTIOCENTRIC, LOCAL_TANGENT_SPACE_EUCLIDEAN, LOCOCENTRIC_EUCLIDEAN_3D, and LOCAL_SPACE_RECTANGULAR_3D.

### 10.5.3 Specification of direction

A *direction* in an orthogonal CS based $\mathrm{SRF_S}$ shall be comprised of:

a) a coordinate $\boldsymbol{c}$ in the interior of the CS domain of $\mathrm{SRF_S}$, and

b) a unit vector $\boldsymbol{n}$ in the local tangent frame at $\boldsymbol{c}$.

---

[28] Not necessarily a direction or a unit vector, but any vector of interest.

[29] All of the 3D SRFTs in this International Standard are based on orthogonal CSs.

The coordinate $\boldsymbol{c}$ is termed the *reference coordinate* of the direction and its corresponding position is termed the *reference position* for the direction. The vector $\boldsymbol{n}$ is termed the *direction vector at* $\boldsymbol{c}$.

NOTE The local tangent frame at a coordinate is an instance of the SRFT LOCOCENTRIC_EUCLIDEAN_3D that provides a vector space setting for vector operations on direction vectors at $\boldsymbol{c}$.

EXAMPLE 1 If $\text{SRF}_\text{S}$ is a LOCOCENTRIC_EUCLIDEAN_3D SRF with SRF parameters $\boldsymbol{q}$, $\boldsymbol{r}$ and $\boldsymbol{s}$, and $\boldsymbol{c}$ is an $\text{SRF}_\text{S}$ reference coordinate, then local tangent vectors at $\boldsymbol{c}$ are equal to the SRF parameters $\boldsymbol{r}$ and $\boldsymbol{s}$. If $\boldsymbol{c} = (0,0,0)$, then $\text{SRF}_\text{L}$ = $\text{SRF}_\text{S}$.

EXAMPLE 2 $\text{SRF}_\text{S}$ is an EQUATORIAL_INERTIAL SRF. This SRF is based on the EQUATORIAL_SPHERICAL CS. If $\boldsymbol{c} = (\lambda_0, \theta_0, \rho_0)$ is a reference coordinate, then the local tangent vectors at $\boldsymbol{c}$ are:

$$\boldsymbol{r} = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} \text{ and } \boldsymbol{s} = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

where:

$$\boldsymbol{v}_1 = \left(\frac{\mathrm{d}\boldsymbol{C}_1}{\mathrm{d}\lambda}\right)_{\lambda=\lambda_0} = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\big(\rho_0 \cos(\theta_0)\cos(\lambda),\ \ \rho_0 \cos(\theta_0)\sin(\lambda),\ \ \rho_0 \sin(\theta_0)\big)\right)_{\lambda=\lambda_0}$$

$$= \big(-\rho_0 \cos(\theta_0)\sin(\lambda_0),\ \ \rho_0 \cos(\theta_0)\cos(\lambda_0),\ \ 0\big),$$

$$\frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \big(-\sin(\lambda_0),\ \ \cos(\lambda_0),\ \ 0\big),$$

$$\boldsymbol{v}_2 = \left(\frac{\mathrm{d}\boldsymbol{C}_2}{\mathrm{d}\theta}\right)_{\theta=\theta_0} = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\big(\rho_0 \cos(\theta)\cos(\lambda_0),\ \ \rho_0 \cos(\theta)\sin(\lambda_0),\ \ \rho_0 \sin(\theta)\big)\right)_{\theta=\theta_0}$$

$$= \big(-\rho_0 \sin(\theta_0)\cos(\lambda_0),\ \ -\rho_0 \sin(\theta_0)\sin(\lambda_0),\ \ \rho_0 \cos(\theta_0)\big), \text{ and}$$

$$\frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \big(-\sin(\theta_0)\cos(\lambda_0),\ \ -\sin(\theta_0)\sin(\lambda_0),\ \ \cos(\theta_0)\big).$$

EXAMPLE 3 $\text{SRF}_\text{S}$ is a CELESTIODETIC SRF. This SRF is based on the GEODETIC CS. If $\boldsymbol{c} = (\lambda_0, \varphi_0, h_0)$ is a reference coordinate, then the local tangent vectors at $\boldsymbol{c}$ are:

$$\boldsymbol{r} = \big(-\sin(\lambda_0),\ \ \cos(\lambda_0),\ \ 0\big),$$

$$\boldsymbol{s} = \big(-\sin(\varphi_0)\cos(\lambda_0),\ \ -\sin(\varphi_0)\sin(\lambda_0),\ \ \cos(\varphi_0)\big), \text{ and}$$

$$\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s} = \big(\cos\lambda_0 \cos\varphi_0,\ \ \sin\lambda_0 \cos\varphi_0,\ \ \sin\varphi_0\big).$$

In this example, $\text{SRF}_\text{L}$ is equivalent to a LOCAL_TANGENT_SPACE_EUCLIDEAN SRF with template parameter values $\lambda = \lambda_0$, $\varphi = \varphi_0$, $\alpha = 0$, $x_F = y_F = 0$, and $h_0$.

EXAMPLE 4 $\text{SRF}_\text{S}$ is based on an augmented conformal map projection CS. If $\boldsymbol{c} = (u_0, v_0, h_0)$ is a reference coordinate, and $(\lambda_0, \varphi_0, h_0)$ is the corresponding celestiodetic coordinate, then the local tangent vectors at $\boldsymbol{c}$ are:

$$\boldsymbol{r} = \big(-\sin\lambda_0 \cos\gamma_0 + \cos\lambda_0 \sin\varphi_0 \sin\gamma_0,\ \ \cos\lambda_0 \cos\gamma_0 + \sin\lambda_0 \sin\varphi_0 \sin\gamma_0,\ \ -\cos\varphi \sin\gamma_0\big), \text{ and}$$

$$\boldsymbol{s} = \big(-\sin\lambda_0 \sin\gamma_0 - \cos\lambda_0 \sin\varphi_0 \cos\gamma_0,\ \ \cos\lambda_0 \sin\gamma_0 - \sin\lambda_0 \sin\varphi_0 \cos\gamma_0,\ \ \cos\varphi_0 \cos\gamma_0\big)$$

where:

$\gamma_0 = \gamma(\lambda_0, \varphi_0)$ the convergence of the meridian.

In this example, $\text{SRF}_\text{L}$ is equivalent to a LOCAL_TANGENT_SPACE_EUCLIDEAN SRF with template parameter values $\lambda = \lambda_0$, $\varphi = \varphi_0$, $\alpha = \gamma_0$, $x_F = y_F = 0$, and $h_0$.

### 10.5.4 Changing the reference coordinate of a direction

Given a direction represented with direction vector $\boldsymbol{n}_1$ at reference coordinate $\boldsymbol{c}_1$, the same direction may be represented at another reference coordinate $\boldsymbol{c}_2$ in the same SRF, with direction vector $\boldsymbol{n}_2$. The direction vector $\boldsymbol{n}_2$ is computed as:

$$\boldsymbol{n}_2 = \boldsymbol{R}\,\boldsymbol{n}_1$$

where:

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{r}_1 \bullet \boldsymbol{r}_2 & \boldsymbol{s}_1 \bullet \boldsymbol{r}_2 & \boldsymbol{t}_1 \bullet \boldsymbol{r}_2 \\ \boldsymbol{r}_1 \bullet \boldsymbol{s}_2 & \boldsymbol{s}_1 \bullet \boldsymbol{s}_2 & \boldsymbol{t}_1 \bullet \boldsymbol{s}_2 \\ \boldsymbol{r}_1 \bullet \boldsymbol{t}_2 & \boldsymbol{s}_1 \bullet \boldsymbol{t}_2 & \boldsymbol{t}_1 \bullet \boldsymbol{t}_2 \end{pmatrix},$$

$\boldsymbol{r}_i$ and $\boldsymbol{s}_i$ are the local tangent vectors at $\boldsymbol{c}_i$, and

$\boldsymbol{t}_i = \boldsymbol{r}_i \times \boldsymbol{s}_i$ for $i = 1, 2$. (10.13)

The local tangent vectors are computed as in Equation (10.12). The matrix $\boldsymbol{R}$ in Equation (10.13) is the direction cosine matrix of the local tangent frame at $\boldsymbol{c}_2$ with respect to the local tangent frame at $\boldsymbol{c}_1$ (see Equation (6.6)).

If the SRF is based on a linear CS, then matrix $\boldsymbol{R}$ is the identity matrix and $\boldsymbol{n}_1 = \boldsymbol{n}_2$. This implies that in a linear orthonormal SRF, a direction vector is independent of the reference coordinate. Thus, Equation (10.13) is only of interest in the case of a curvilinear SRF.

### 10.5.5 Representing a direction in a different SRF

Given a direction represented with direction vector $\boldsymbol{n}_{\mathrm{S}}$ at $\boldsymbol{c}_{\mathrm{S}}$ in SRF$_{\mathrm{S}}$, the same direction may be represented at reference coordinate $\boldsymbol{c}_{\mathrm{T}}$, with direction vector $\boldsymbol{n}_{\mathrm{T}}$ in SRF$_{\mathrm{T}}$. If $\boldsymbol{H}_{\mathrm{ST}}$ is the similarity transformation from ORM$_{\mathrm{S}}$ to ORM$_{\mathrm{T}}$ and $\boldsymbol{M}_{\mathrm{ST}}$ is the matrix in the last term in Equation (10.3), then the direction vector $\boldsymbol{n}_{\mathrm{T}}$ is computed as:

$$\boldsymbol{n}_{\mathrm{T}} = \boldsymbol{R}_{\mathrm{ST}}\,\boldsymbol{n}_{\mathrm{S}}$$

where:

$\boldsymbol{R}_{\mathrm{ST}} = \boldsymbol{R}_{\mathrm{T}}^{\mathrm{T}} \circ \boldsymbol{M}_{\mathrm{ST}} \circ \boldsymbol{R}_{\mathrm{S}}$,

$\boldsymbol{R}_{\mathrm{T}}^{\mathrm{T}}$ = the transpose of $\boldsymbol{R}_{\mathrm{T}}$, and

for: $i$ = S or T,

$$\boldsymbol{R}_i = \begin{pmatrix} r_{i,1} & s_{i,1} & t_{i,1} \\ r_{i,2} & s_{i,2} & t_{i,2} \\ r_{i,3} & s_{i,3} & t_{i,3} \end{pmatrix},$$

$\boldsymbol{t}_i = \boldsymbol{r}_i \times \boldsymbol{s}_i = \begin{pmatrix} t_{i,1} & t_{i,2} & t_{i,3} \end{pmatrix}$,

$\boldsymbol{r}_i = \begin{pmatrix} r_{i,1} & r_{i,2} & r_{i,3} \end{pmatrix}$ and $\boldsymbol{s}_i = \begin{pmatrix} s_{i,1} & s_{i,2} & s_{i,3} \end{pmatrix}$ are the local tangent vectors at $\boldsymbol{c}_i$. (10.14)

Equation (10.14) is derived from Equation (10.9) by dropping the translation term since directions are translation invariant, and dropping the scale factor $(1+\Delta s_{\mathrm{SR}})/(1+\Delta s_{\mathrm{TR}})$ since $\boldsymbol{n}_{\mathrm{T}}$ is a unit vector.

The rotation matrix $\boldsymbol{R}_{\mathrm{ST}}$ in Equation 10.14 is termed the *orientation of SRF$_{\mathrm{T}}$ at reference point $\boldsymbol{c}_{\mathrm{T}}$, with respect to SRF$_{\mathrm{S}}$ at reference point $\boldsymbol{c}_{\mathrm{S}}$*. The rotation matrix $\boldsymbol{R}_{\mathrm{ST}}$ is a generalization of the matrix in Equation (10.13) that accounts for the change of position-space between the source and target ORMs.

EXAMPLE 1 SRF$_{\mathrm{S}}$ is SRF GEODETIC_WGS_1984 and SRF$_{\mathrm{T}}$ is SRF GEOCENTRIC_WGS_1984. With SRF$_{\mathrm{S}}$ reference coordinate $\boldsymbol{c}_{\mathrm{S}} = (\lambda,\ \varphi,\ h) = (-77\,\pi/180,\ +38{,}88\,\pi/180,\ 0)$, the Washington monument, an obelisk, points approximately in the direction $\boldsymbol{n}_{\mathrm{S}} = (0,\ 0,\ 1)$ at $\boldsymbol{c}_{\mathrm{S}}$. In this example, ORM$_{\mathrm{S}}$ = ORM$_{\mathrm{T}}$ so that $\boldsymbol{M}_{\mathrm{ST}}$ is the identity matrix, and because SRF$_{\mathrm{T}}$ is based on SRFT CELESTIOCENTRIC, $\boldsymbol{R}_{\mathrm{T}}$ is also the identity matrix. Consequently Equation (10.14) reduces to:

$$\boldsymbol{n}_\mathrm{T} = \boldsymbol{R}_\mathrm{S}\boldsymbol{n}_\mathrm{S} = \begin{pmatrix} r_{\mathrm{S},1} & s_{\mathrm{S},1} & t_{\mathrm{S},1} \\ r_{\mathrm{S},2} & s_{\mathrm{S},2} & t_{\mathrm{S},2} \\ r_{\mathrm{S},3} & s_{\mathrm{S},3} & t_{\mathrm{S},3} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{t}_\mathrm{S}.$$

Then using the expression in 10.5.3 Example 3 for $\boldsymbol{t}$:

$$\begin{aligned}\boldsymbol{t}_\mathrm{S} &= \left(\cos\lambda_0\cos\varphi_0 \quad \sin\lambda_0\cos\varphi_0 \quad \sin\varphi_0\right) \\ &= \left(\cos(-77\,\pi/180)\cos(38{,}88\,\pi/180) \quad \sin(-77\,\pi/180)\cos(38{,}88\,\pi/180) \quad \sin(38{,}88\,\pi/180)\right) \\ &= (0{,}175\ 115\ 92 \quad -0{,}758\ 510\ 36 \quad 0{,}627\ 691\ 36).\end{aligned}$$

The resulting vector $\boldsymbol{n}_\mathrm{T} = (0{,}175\ 115\ 92 \quad -0{,}758\ 510\ 36 \quad 0{,}627\ 691\ 36)$ is the direction vector in $\mathrm{SRF_T}$.

Engineering designs and abstract models are often intended for realization in the physical world. In such cases, the operation of changing the representation of direction vector $\boldsymbol{n}_\mathrm{S}$ in a linear SRF representing the abstract space to a direction vector $\boldsymbol{n}_\mathrm{T}$ in a linear SRF representing the physical object space is based on Equation (10.11). In the notation of 10.4.6:

$$\boldsymbol{n}_\mathrm{T} = \frac{1}{|\boldsymbol{W}|}\boldsymbol{W} \circ \boldsymbol{R}_\mathrm{S}\boldsymbol{n}_\mathrm{S}\,. \qquad (10.15)$$

Division by the determinant cancels any scaling by matrix $\boldsymbol{W}$ to ensure that $\boldsymbol{n}_\mathrm{T}$ is a unit vector. (The rotation matrix $\boldsymbol{R}_\mathrm{S}$ does not change the length of $\boldsymbol{n}_\mathrm{S}$.)

EXAMPLE 2 In ISO/IEC 18023-1, if an instance of the class <DRM Geometry Model Instance> has a component of class <DRM World Transformation>, that component specifies an invertible matrix $\boldsymbol{W}$ and a coordinate $\boldsymbol{c}$ in the <DRM Environment Root> SRF. If $\boldsymbol{c}_\mathrm{S}$ and $\boldsymbol{n}_\mathrm{S}$ are a reference coordinate and a direction vector in an associated LOCAL_SPACE_RECTANGULAR_3D <DRM Geometry Model>, and $\mathrm{SRF_T}$ is the local tangent frame at $\boldsymbol{c}$, then Equation (10.11) and Equation (10.15) may be used to compute $\boldsymbol{c}_\mathrm{T}$ and $\boldsymbol{n}_\mathrm{T}$, respectively. The methods of 10.4.3 may be used to further change $\boldsymbol{c}_\mathrm{T}$ from $\mathrm{SRF_T}$ to the <DRM Environment Root> SRF. This procedure to change <DRM Geometry Model> coordinates and directions to the environment root SRF is termed "model instancing".

### 10.5.6 Representing a vector quantity in a different SRF

Vectors combine a direction with a magnitude, and are used to describe a number of properties of moving objects, such as their velocities, accelerations and other quantities. Similar to directions, vectors are defined with respect to the axes of a specific $\mathrm{SRF_L}$ at a reference point. All properties and operations that apply to directions also apply to vectors.

Given a vector quantity $\boldsymbol{v}_\mathrm{S}$ at reference location $\boldsymbol{c}_\mathrm{S}$ in $\mathrm{SRF_S}$, it may be represented as $\boldsymbol{v}_\mathrm{T}$ at reference location $\boldsymbol{c}_\mathrm{T}$ in $\mathrm{SRF_T}$. If $\boldsymbol{R}_\mathrm{ST}$ is the orientation $\mathrm{SRF_S}$ at reference point $\boldsymbol{c}_\mathrm{S}$, with respect to $\mathrm{SRF_T}$ at reference point $\boldsymbol{c}_\mathrm{T}$, the vector $\boldsymbol{v}_\mathrm{T}$ is computed as:

$$v_\mathrm{T} = \boldsymbol{R}_\mathrm{ST}v_\mathrm{S}$$

Given a vector quantity $\boldsymbol{v}_\mathrm{B}$ in a body frame $\mathrm{SRF_B}$ (or in general any linear orthonormal reference frame) whose orientation at a reference location $\boldsymbol{c}_\mathrm{S}$ in $\mathrm{SRF_S}$ is known, that vector quantity may be represented as $\boldsymbol{v}_\mathrm{T}$ at reference location $\boldsymbol{c}_\mathrm{T}$ in $\mathrm{SRF_T}$. If $\boldsymbol{R}_\mathrm{BS}$ is the orientation of $\mathrm{SRF_B}$ at reference point $\boldsymbol{c}_\mathrm{S}$, with respect to $\mathrm{SRF_S}$, the vector $\boldsymbol{v}_\mathrm{T}$ is computed as: $v_\mathrm{T} = \boldsymbol{R}_\mathrm{ST} \circ \boldsymbol{R}_\mathrm{BS}v_\mathrm{B}$.

### 10.5.7 Representing an orientation in a different SRF

The orientation of an object E in 3D space specifies how a set of orthogonal axes attached to that object are aligned with respect to the axes of a specific orthogonal SRF (see 6.1). The orientation of an object specifies a rotation operation that would bring the SRF axes into alignment with the corresponding object axes (or vice versa).

As with directions, orientations that are specified with respect to an SRF use the unique local tangent frame $SRF_L$ at a specified reference location (see 10.5.2).

Given $\boldsymbol{R}_{ES}$, the orientation of an object E at reference location $\boldsymbol{c}_S$ with respect to $SRF_S$, the orientation of that object may be represented as $\boldsymbol{R}_{ET}$, at reference location $\boldsymbol{c}_T$, with respect to $SRF_T$. If $\boldsymbol{R}_{ST}$ is the orientation of the $SRF_S$ at reference point $\boldsymbol{c}_S$, with respect to $SRF_T$ at reference point $\boldsymbol{c}_T$, the orientation $\boldsymbol{R}_{ET}$ is computed as: $\boldsymbol{R}_{ET} = \boldsymbol{R}_{ST} \circ \boldsymbol{R}_{ES}$ (see Equation (6.2)).

## 10.6 Euclidean distance

This International Standard supports an operation to return the Euclidean distance between two object-space locations using the coordinates of those locations in an SRF.

If $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are two coordinates in an SRF, and if $\boldsymbol{G}$ is the generating function of the CS of the SRF, the *Euclidean distance* $d_E$ between the corresponding points in object-space is given by:

$$d_E(\boldsymbol{c}_1, \boldsymbol{c}_2) = d\big(\boldsymbol{G}(\boldsymbol{c}_1), \boldsymbol{G}(\boldsymbol{c}_2)\big)$$

where $d$ is the Euclidean metric.

## 10.7 Geodesic distance operations

### 10.7.1 Introduction

A curve on a smooth surface that has the property that any sufficiently small segment of it realizes the shortest distance on the surface between the segment's two endpoints is termed a geodesic. The formal definition of a geodesic is given in A.7.4.

EXAMPLE 1 On a sphere, the equator, the meridians, and all other great circles are geodesics. Likewise any segment of one of these curves is a geodesic. No parallel of latitude except the equator is a geodesic.

EXAMPLE 2 On an oblate ellipsoid, the equator is a geodesic, and the meridians are all geodesics. All the other geodesics are curves which cross the equator at some non-right angle and wind around the ellipsoid between two parallels of opposite latitude (see Figure 10.7).

Let points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ lie on a smooth surface. The shortest distance on the surface from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ is the shortest arc length associated with any of the smooth surface curves that connect $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$. This distance is unique, but the curve that has this arc length may not be unique. In particular, for the two pole points, every meridian is such a curve.

EXAMPLE 3 On an oblate ellipsoid, let $\boldsymbol{p}_1$ be the point with surface geodetic coordinates (λ, φ) = (0°,20°) and let $\boldsymbol{p}_2$ be the point diametrically opposite, *i.e.,* with surface geodetic coordinates (λ, φ) = (180°,-20°). Then the shortest distance on the surface from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ is twice the meridional quadrant, *i.e.*, twice the length of a meridian from equator to pole. But there are two distinct curves from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ which have this number as their arc length – one passes through the north pole and the other passes through the south pole. (Both are composed of segments of meridians).

EXAMPLE 4 On an oblate ellipsoid with eccentricity $\varepsilon$, let points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ lie on the equator but be separated by a longitude difference that is less than $\pi$ and more than $\pi\sqrt{1-\varepsilon^2}$, an angle termed the "lift-off longitude". Then there will be two curves from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ whose arc length is the shortest distance from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ – one lying in the northern hemisphere, the other lying (symmetrically) in the southern hemisphere.

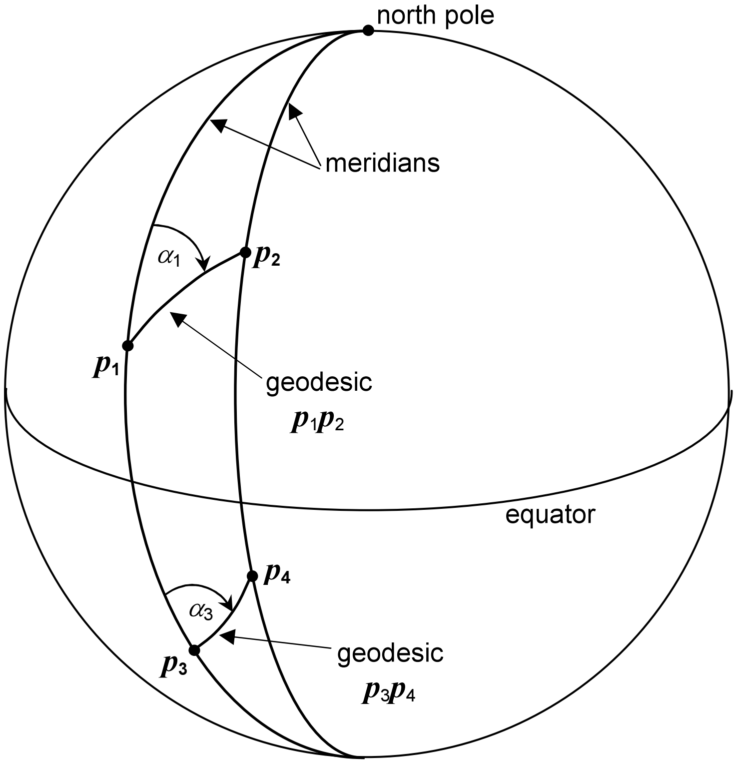


**Figure 10.7 — Examples of geodesics**

If a curve lying on a smooth surface connects point $p_1$ to point $p_2$, and if that curve's arc length is also the shortest distance from $p_1$ to $p_2$, then that curve is a geodesic. Thus, the arc length of the shortest curve connecting the two points is termed the *geodesic distance.*

EXAMPLE 5 The two curves from $\mathbf{p}_1$ to $\mathbf{p}_2$ defined in Example 3 are geodesics.

EXAMPLE 6 The two curves from $\mathbf{p}_1$ to $\mathbf{p}_2$ defined in Example 4 are geodesics.

The converse is not true. If a geodesic starts at point $p_1$ and ends at point $p_2$, its arc length may or may not be the same as the shortest distance on the surface from $p_1$ to $p_2$.

EXAMPLE 7 Let points $p_1$ and $p_2$ lie on the equator of a sphere or oblate ellipsoid at longitudes 0° and 181°, respectively. The segment of the equator from $p_1$ to $p_2$ that is continuous in longitude from 0° to 181° is a geodesic. (All segments of the equator are geodesics). However, its arc length is not the shortest distance on the surface from $p_1$ to $p_2$. Any curve which realizes the shortest distance on the surface from $p_1$ to $p_2$ has to lie within a single hemisphere of longitude.

There are two problems of interest pertaining to geodesics on an oblate ellipsoid. In the first, termed the direct problem, a surface point, an azimuth, and a distance are given. The problem is to find a second surface point which terminates the (unique) geodesic whose initial point is the given point, whose initial forward azimuth is the given azimuth, and whose arc length is the given distance. Also to be found is the geodesic's terminal forward azimuth. The details are given in 10.7.3.

In the second problem, termed the indirect problem, two distinct surface points are given. The problem is to find the shortest distance on the surface between the two given points, and find the set of curves (which will be geodesics) whose arc lengths equal this shortest distance. In addition, the initial and terminal forward azimuths of each curve is to be found. The details are in 10.7.4.

This International Standard supports the geodesic operations for SRFs based on SRFT CELESTIODETIC, PLANETODETIC, and all map projection SRFTs.

 

Given two surface coordinates $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ of points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, respectively, the *geodesic distance operation*:

$$s = d_G\left(\boldsymbol{c}_1, \boldsymbol{c}_2\right)$$

is defined as the indirect problem for ($\lambda_1$, $\varphi_1$) and ($\lambda_2$, $\varphi_2$) where ($\lambda_1$, $\varphi_1$) is the surface geodetic coordinate for $\boldsymbol{c}_1$ and ($\lambda_2$, $\varphi_2$) is the surface geodetic coordinate for $\boldsymbol{c}_2$.

An extended version of this operation provides the forward azimuth value $\alpha_1$ at $\boldsymbol{c}_1$ and the forward azimuth value $\alpha_2$ at $\boldsymbol{c}_2$:

$$\left\{s, \alpha_1, \alpha_2\right\} = d_{GI}\left(\boldsymbol{c}_1, \boldsymbol{c}_2\right).$$

The geodesic destination operation requires a starting point $\boldsymbol{c}_1$, a forward azimuth value $\alpha_1$, at $\boldsymbol{c}_1$ and a positive distance $s$. It returns the destination point $\boldsymbol{c}_2$ and the forward azimuth value $\alpha_2$ at $\boldsymbol{c}_2$:

$$\left\{\boldsymbol{c}_2, \alpha_2\right\} = d_{GD}\left(\boldsymbol{c}_1, \alpha_1, s\right)$$

where { ($\lambda_2$, $\varphi_2$), $\alpha_2$} is the direct problem solution for input parameter values {($\lambda_1$, $\varphi_1$), $\alpha_1$, $s$}.

There is a large body of literature concerning computational techniques to solve the direct and indirect problems. In the interest of accuracy and computational efficiency, many of these computational techniques treat the problems by sub-cases -- short lines, long lines, intermediate length lines, and other caveats and exceptions. Each of these has been optimized in a way that is appropriate for the intended application or user domain. For purposes of this International Standard, a recently published treatment ([ROL10]) that has one mathematical formulation to cover all cases is utilized.

### 10.7.2 Auxiliary functions

The treatment of the direct and indirect problems in 10.7.3 and 10.7.4 require the auxiliary functions defined in this subclause.

An important characteristic of a geodesic on an oblate ellipsoid is that the quantity termed the (non-metric) *Clairaut constant* and defined by:

$$c = \frac{\sin(\alpha)\cos(\varphi)}{\sqrt{1-\varepsilon^2 \sin^2(\varphi)}}$$

has a constant value at every point on a given geodesic, where ($\lambda$, $\varphi$) is the coordinate of a point on the geodesic and $\alpha$ is the azimuth of the curve at that point.

The mathematics required to solve the direct and indirect problems involves the use of elliptic integrals. The incomplete elliptic integral of third kind is defined for real $n, \theta,$ and $m,$ with $m^2 < 1$ as:

$$P(n,\theta,m) = \int_0^\theta \frac{d\xi}{\left(1-n\sin^2\xi\right)\sqrt{1-m\sin^2\xi}}.$$

The treatment in [ROL10] defines two auxiliary functions: a longitude difference function $\mathrm{L}\left(c,\theta_1,\theta_2\right)$ and an arc length function $\mathrm{A}\left(c,\theta_1,\theta_2\right)$ that are defined for all values of $c$, $\theta_1$ and $\theta_2$ by:

$$\mathrm{L}(c,\theta_1,\theta_2) = \frac{c(1-\varepsilon^2)}{\sqrt{1-c^2\varepsilon^2}}\left(P(k^2,\theta_2,k^2\varepsilon^2) - P(k^2,\theta_1,k^2\varepsilon^2)\right),\ c \neq 0 .$$
$$\mathrm{L}(0,\theta_1,\theta_2) = \lim_{c\to 0^+} \mathrm{L}(c,\theta_1,\theta_2) \quad (10.16)$$

and

$$\mathrm{A}(c,\theta_1,\theta_2) = \frac{a(1-\varepsilon^2)}{\sqrt{1-c^2\varepsilon^2}}\left(P(k^2\varepsilon^2,\theta_2,k^2\varepsilon^2) - P(k^2\varepsilon^2,\theta_1,k^2\varepsilon^2)\right). \quad (10.17)$$

where

$$k^2 = \frac{1-c^2}{1-c^2\varepsilon^2}.$$

### 10.7.3 The direct problem

Given an oblate ellipsoid with major semi-axis $a$ and eccentricity $\varepsilon$ , let $\boldsymbol{p}_1$ be a non-polar point on the ellipsoid given by its surface geodetic coordinates ($\lambda_1$, $\varphi_1$). Let a geodesic be defined with $\boldsymbol{p}_1$ as its initial point, $\alpha_1$ as its initial forward azimuth, and arc length $s$ . This geodesic will terminate at a point $\boldsymbol{p}_2$.

The direct problem requires finding the surface geodetic coordinates ($\lambda_2$, $\varphi_2$) of $\boldsymbol{p}_2$ and the forward azimuth $\alpha_2$ of the geodesic at the point $\boldsymbol{p}_2$. The quantity $\alpha_2 + \pi$ is termed the *back azimuth* at $\boldsymbol{p}_2$ as it points backwards toward $\boldsymbol{p}_1$.

The given parameters are restricted to $-\pi/2 < \varphi_1 < \pi/2$, $-\pi < \alpha_1 \leq \pi$, and $s > 0$.

The functions $\mathrm{L}(c,\theta_1,\theta_2)$ and $\mathrm{A}(c,\theta_1,\theta_2)$ are used to solve the direct problem.

The given values in the direct problem ($\lambda_1$, $\varphi_1$) and $\alpha_1$ determine $c$,

$$c = \frac{\sin(\alpha_1)\cos(\varphi_1)}{\sqrt{1-\varepsilon^2\sin^2(\varphi_1)}}.$$

Then,

$$\lambda_2 = \lambda_1 + \mathrm{L}(c,\theta_1,\theta_2),$$
$$\varphi_2 = \arcsin(k\sin\theta_2),\ \text{and}$$
$$\alpha_2 = \operatorname{arctan2}\left(c\sqrt{1-k^2\sin\theta_2}, k\sqrt{1-k^2\varepsilon^2\sin\theta_2}\right)$$

where

$$\theta_1 = \arcsin(\sin(\varphi_1)/k),\ k = \pm\sqrt{\frac{1-c^2}{1-c^2\varepsilon^2}},\ k \geq 0 \text{ if } |\alpha_1| \leq \frac{\pi}{2} \text{ and } k < 0 \text{ otherwise, and}$$

$\theta_2$ is determined by:

$$s = \mathrm{A}(c,\theta_1,\theta_2). \quad (10.18)$$

 

Equation 10.18 has a unique solution for $\theta_2$. Reference [ROL12] gives the following Newton-Raphson iteration, which rapidly converges to the solution:

$$\theta_2^{[1]} = \theta_1 + \frac{\pi s}{\mathrm{A}(c, 0, \pi)},$$

$$\theta_2^{[n+1]} = \theta_2^{[n]} - \left(\frac{\sqrt{1-c^2\varepsilon^2}}{a\left(1-\varepsilon^2\right)}\right)\left(1-k^2\varepsilon^2 \sin\theta_2^{[n]}\right)^{3/2}\left(\mathrm{A}\left(c,\theta_1,\theta_2^{[n]}\right)-s\right).$$

### 10.7.4 The indirect problem

Given an oblate ellipsoid with major semi-axis $a$ and eccentricity $\varepsilon$, let $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ be two points on the ellipsoid given by their surface geodetic coordinates ($\lambda_1$, $\varphi_1$) and ($\lambda_2$, $\varphi_2$).

The indirect problem requires finding the shortest distance $s$ on the ellipsoid from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$. Further, for each curve from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ whose arc length is $s$, it is required to find the forward azimuths $\alpha_1$ and $\alpha_2$ at the points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ respectively. (Such curves will be geodesics, and there will be 1, 2, or infinitely many of them.)

The given parameters are restricted to $-\pi \le \lambda_2 - \lambda_1 \le \pi$, $-\pi/2 \le \varphi_1 \le \pi/2$, and $-\pi/2 \le \varphi_2 \le \pi/2$.

The solution to the indirect problem can be determined once $c$, the Clairaut constant for the solution geodesic curve segment, is found. Dealing with the extreme $c$ values 0 and 1 separately simplifies the process.

<u>The single meridional case</u>: $c$ = 0 if $\lambda_2 = \lambda_1$ or if either point is a pole ($|\varphi_1| = \frac{\pi}{2}$ or $|\varphi_2| = \frac{\pi}{2}$). Then if $\varphi_1 < \varphi_2$, the solution is:

$$s = \mathrm{A}(0, \varphi_1, \varphi_2), \text{ and } \alpha_1 = \alpha_2 = 0.$$

Otherwise $\phi_1 > \phi_2$, and the solution is:

$$s = \mathrm{A}(0, \varphi_2, \varphi_1), \text{ and } \alpha_1 = \alpha_2 = \pi.$$

If either point is a pole, the azimuth at that point is undefined. The solution geodesic curve segment is unique unless both given points are poles. In that case the solution set is the infinite set of all meridians.

<u>Meridional segments joined at pole</u>: $c$ = 0 if $\lambda_2 = \lambda_1 \pm \pi$ and $\varphi_2 \ge -\varphi_1$. Then

$$s = \mathrm{A}(0, \varphi_1, \pi - \varphi_2), \ \alpha_1 = 0, \alpha_2 = \pi$$

and the geodesic curve segment passes through the north pole.

Similarly, $c$ = 0 if $\lambda_2 = \lambda_1 \pm \pi$ and $\varphi_2 < -\varphi_1$. Then

$s = \mathrm{A}(0, \varphi_1, -\varphi_2 - \pi)$, $\alpha_1 = \pi$, $\alpha_2 = 0$ and the geodesic curve segment passes through the south pole.

<u>Equatorial segment</u>: $c$ = 1 if $\varphi_1 = \varphi_2 = 0$ and $0 < |\lambda_2 - \lambda_1| \le \pi\sqrt{1-\varepsilon^2}$. Then

$$s = a|\lambda_2 - \lambda_1|, \text{ and } \alpha_1 = \alpha_2 = \tfrac{\pi}{2} \text{ if } \alpha_1 < \alpha_2 \text{ and } \alpha_1 = \alpha_2 = -\tfrac{\pi}{2} \text{ otherwise.}$$

The solution is unique.

Near-antipodal equatorial segment: If $\varphi_1 = \varphi_2 = 0$ and the points are separated by more than the lift-off longitude ($\pi\sqrt{1-\varepsilon^2} < |\lambda_2 - \lambda_1| < \pi$), then $c$ is determined by solving the equation:

$\lambda_2 - \lambda_1 = \mathrm{L}(c, 0, \pi)$ in the interval $0 \le c \le 1$.

Assuming $\lambda_1 < \lambda_2$, the solution parameters are then given by:

$s = \mathrm{A}(c, 0, \pi)$, $\alpha_1 = \arcsin(c)$, and $\alpha_2 = \pi - \alpha_1$.

This geodesic curve segment lies in the northern hemisphere. A second solution lies in the southern hemisphere in north-south symmetry.

Prograde typical: The remaining cases may be reduced to the prograde typical case of $0 < \lambda_2 - \lambda_1 < \pi$ and $\varphi_2 \ge |\varphi_1| \ne 0$.

Define $c_{\max} = \dfrac{\cos\varphi_2}{\sqrt{1-\varepsilon^2 \sin^2 \varphi_2}}$ and $\lambda_{\text{crit}} = L\left(c_{\max}, \left(\dfrac{\sin\varphi_1}{\sin\varphi_2}\right), \dfrac{\pi}{2}\right)$.

Then $c$ may be determined by an iterative solution of the equation:

$\lambda_2 - \lambda_1 = \mathrm{L}(c, \theta_1(c), \theta_2(c))$ in the interval $0 \le c \le c_{\max}$,

where

$\theta_1(c) = \arcsin(\sin(\varphi_1)/k(c))$, $k(c) = \sqrt{\dfrac{1-c^2}{1-c^2\varepsilon^2}}$, and

$$\theta_1(c) = \begin{cases} \arcsin(\sin(\varphi_1)/k(c)), & \text{if } \lambda_2 - \lambda_1 < \lambda_{\text{crit}} \\ \pi/2, & \text{if } \lambda_2 - \lambda_1 = \lambda_{\text{crit}} \\ \pi - \arcsin(\sin(\varphi_1)/k(c)), & \text{if } \lambda_2 - \lambda_1 > \lambda_{\text{crit}} \end{cases}.$$

The solution parameters are determined by $c$:

$$s = \mathrm{A}(c, \theta_1(c), \theta_2(c)),$$
$$\alpha_1 = \operatorname{arctan2}\left(c\sqrt{1-\varepsilon^2 \sin\varphi_1}, k(c)\sqrt{1-c^2}\cos\varphi_1\right), \text{ and}$$
$$\alpha_2 = \operatorname{arctan2}\left(c\sqrt{1-\varepsilon^2 \sin\varphi_2}, k(c)\sqrt{1-c^2}\cos\varphi_2\right).$$

NOTE Extremely small values of $c$ can cause numerical instability in some implementations. Alternative methods to evaluate $\mathrm{L}(c, \theta_1, \theta_2)$ in this and other difficult cases are treated in [ROL12].

Other prograde cases: If $0 < \lambda_2 - \lambda_1 < \pi$ and cases above do not apply, a new pair of points $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$ that satisfy the prototypical case constraints can be specified using parameters from the given pair $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. The indirect problem solution for points $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$, the shortest distance between them $\tilde{s}$, and the forward azimuths $\alpha_3$ and $\alpha_4$ will determine the solution for $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ as follows:

 

If $|\varphi_2| \le \varphi_1$, let $\boldsymbol{p}_3 = (\lambda_1, \varphi_2)$ and $\boldsymbol{p}_4 = (\lambda_2, \varphi_1)$. Then $\alpha_1 = \pi - \alpha_4$ and $\alpha_2 = \pi - \alpha_3$.

If $|\varphi_2| \le -\varphi_1$, let $\boldsymbol{p}_3 = (\lambda_1, -\varphi_2)$ and $\boldsymbol{p}_4 = (\lambda_2, -\varphi_1)$. Then $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$.

If $|\varphi_1| \le -\varphi_2$, let $\boldsymbol{p}_3 = (\lambda_1, -\varphi_1)$ and $\boldsymbol{p}_4 = (\lambda_2, -\varphi_2)$. Then $\alpha_1 = \pi - \alpha_3$ and $\alpha_2 = \pi - \alpha_4$.

In all these cases the arc length solution is the same, $s = \tilde{s}$, and the value of $c$ and the multiplicity of shortest geodesic segments are also the same.

Retrograde cases: A retrograde case, $\lambda_2 < \lambda_1$, is converted to a prograde case with $\boldsymbol{p}_3 = (\lambda_2, \varphi_1)$ and $\boldsymbol{p}_4 = (\lambda_1, \varphi_2)$. Then $\alpha_1 = -\alpha_3$, $\alpha_2 = -\alpha_4$, and $s = \tilde{s}$. The value $-c$ from prograde case is the retrograde solution value for $c$ and the multiplicity of shortest geodesic segments are the same.

http://standards.iso.org/ittf/PubliclyAvailableStandards/