6 Orientation

6.1 Introduction

The orientation of an object in space specifies how that object is aligned with respect to a reference configuration of that object. The reference configuration is a conceptual copy of the object that is positioned with respect to a particular spatial reference frame. The orientation of the object may be specified by a length-preserving transformation that would make the reference configuration congruent to the object. Only the rotational components of this transformation are essential for the specification, as translation operations do not affect alignment.

For computational purposes, an orthonormal set of axes is created and attached to the object. These axes are termed the *object axes*. Another orthonormal set of axes is created and attached in the same manner to the corresponding position of the reference configuration. These axes are termed the *reference axes*. An orientation specification is a rotation operation that would bring the reference axes into alignment with the corresponding object axes. Only a single rotation is required for such a specification, since, as a consequence of Euler's rotation theorem, a series of rotations about various axes is equivalent to one rotation about a single axis.

Rotation operator concepts and various mathematical representations of rotations have been in wide use from before the time of Euler's work on the subject. As a result, there are many different treatments in the literature, using similar terms with different meanings and different notational conventions. For this reason, rotation terms and notation used in this International Standard are fully defined.

The specification of an ORM (see 7.4.4) depends on a similarity transformation (see 7.3.2) for which a rotation operator is a key component. Converting the representation of such rotation operators to and from the Matrix representation (see 6.7.2) is required for some change of SRF operations (see 10.3.2 and 10.4.5). Rotation operators are also important in some of the application domains that fall within the scope of this International Standard. This includes the ability to convert an object's orientation represented with respect to one SRF to its equivalent with respect to another SRF.

6.2 Change of coordinate basis and rotations

Three-dimensional Euclidean space forms a vector space once an origin point has been selected. Each Euclidean space point is then associated with the vector that points from the origin to the point and has length equal to the Euclidean distance between the origin and the point. Thus points in space and vectors with respect to a selected origin may be treated as equivalent concepts. This vector space may also be represented by the vector space of three-tuples of scalars provided an orthonormal basis of three vectors centred at the origin is selected. The selection of origin and orthonormal basis is termed an *orthonormal frame*. Every point in three-dimensional Euclidean space is uniquely represented by a linear combination of the basis vectors in an orthonormal frame. The three scalars in the linear combination are represented by a three-tuple in the vector space.

The notion of orientation is translation independent so in this clause, without loss of generality, two or more orthonormal frames may be assumed to have a common origin point. Similarly, a rotation of space about an arbitrary axis line in space is, with translations, equivalent to a rotation about an axis that passes through a designated origin. Select a point on the arbitrary axis and set $\boldsymbol{p}$ to the vector from the origin to the selected point. Translate the axis by $-\boldsymbol{p}$, rotate about the translated axis (which passes through the origin) and finally translate back by $\boldsymbol{p}$. This sequence of operations produces the same result as the rotation about the arbitrary axis.

If $\boldsymbol{E}$ with orthonormal basis $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, and $\boldsymbol{F}$ with orthonormal basis $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}$, are two orthonormal frames with common origin, the coordinate representations of a point $\boldsymbol{r}$ in each frame is given by:

$(r_1, r_2, r_3)_E$, where $\boldsymbol{r} = r_1\boldsymbol{x} + r_2\boldsymbol{y} + r_3\boldsymbol{z}$, and
$(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)_F$, where $\boldsymbol{r} = \tilde{r}_1\tilde{\boldsymbol{x}} + \tilde{r}_2\tilde{\boldsymbol{y}} + \tilde{r}_3\tilde{\boldsymbol{z}}$.

Since each basis is orthonormal, the coordinate-component scalars may be computed as the dot product of $\boldsymbol{r}$ with each corresponding basis vector:

$r_1\boldsymbol{x} + r_2\boldsymbol{y} + r_3\boldsymbol{z} = (\boldsymbol{r} \bullet \boldsymbol{x})\boldsymbol{x} + (\boldsymbol{r} \bullet \boldsymbol{y})\boldsymbol{y} + (\boldsymbol{r} \bullet \boldsymbol{z})\boldsymbol{z}$, and
$\tilde{r}_1\tilde{\boldsymbol{x}} + \tilde{r}_2\tilde{\boldsymbol{y}} + \tilde{r}_3\tilde{\boldsymbol{z}} = (\boldsymbol{r} \bullet \tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} + (\boldsymbol{r} \bullet \tilde{\boldsymbol{y}})\tilde{\boldsymbol{y}} + (\boldsymbol{r} \bullet \tilde{\boldsymbol{z}})\tilde{\boldsymbol{z}}$.

The change coordinate basis operation taking an $\boldsymbol{E}$ coordinate $(r_1, r_2, r_3)_E$ to an $\boldsymbol{F}$ coordinate $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)_F$ shall be denoted by $\boldsymbol{\Omega}_{F \leftarrow E}$. This operation, $\boldsymbol{\Omega}_{F \leftarrow E}$, is a linear transformation and can thus be realized as a matrix multiplication of coordinate column vectors:

$$(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)_F = \boldsymbol{\Omega}_{F \leftarrow E}\left((r_1, r_2, r_3)_E\right)$$

where:

$$\begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}, \text{ and}$$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{x} \bullet \tilde{\boldsymbol{x}} & \boldsymbol{y} \bullet \tilde{\boldsymbol{x}} & \boldsymbol{z} \bullet \tilde{\boldsymbol{x}} \\ \boldsymbol{x} \bullet \tilde{\boldsymbol{y}} & \boldsymbol{y} \bullet \tilde{\boldsymbol{y}} & \boldsymbol{z} \bullet \tilde{\boldsymbol{y}} \\ \boldsymbol{x} \bullet \tilde{\boldsymbol{z}} & \boldsymbol{y} \bullet \tilde{\boldsymbol{z}} & \boldsymbol{z} \bullet \tilde{\boldsymbol{z}} \end{bmatrix} \tag{6.1}$$

Since basis vectors are unit vectors, each dot product in Equation (6.1) is the cosine of the angle between the two vectors (see A.2). This matrix is thus termed the *direction cosine matrix*. Note that the columns of the matrix are the $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ basis vectors in $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}$ coordinate representation while the rows (or columns of the transpose matrix) are the $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}$ basis vectors in $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ coordinate representation. In particular, the transpose $\boldsymbol{M}^{\mathrm{T}}$ is the matrix for the inverse change of coordinate basis operation $(r_1, r_2, r_3)_E = \boldsymbol{\Omega}_{E \leftarrow F}\left((\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)_F\right)$. Thus $\boldsymbol{M}^{-1} = \boldsymbol{M}^{\mathrm{T}}$.

Euler's rotation theorem states that any length-preserving transformation of 3D space that has at least one point fixed under the transformation is equivalent to a single rotation about an axis through that point. If the axis is assigned a direction, the angle of rotation can be specified as a positive angle or a negative angle using the *right-hand rule*: conceptually, if the right-hand holds the axis with thumb pointing in the axis direction, the fingers curl in the positive angle direction.

Euler's rotation theorem applies to any linear transformation that is length preserving, and thus, the transformation has a unit eigenvector $\boldsymbol{n}$ and three eigenvalues: $1, e^{+i\theta}$, and $e^{-i\theta}$. The line passing through the origin and $\boldsymbol{n}$ is fixed under the transformation and represents the axis of rotation. The angle of rotation is given by $\pm\theta$. $\boldsymbol{R}_{\boldsymbol{n}}(\theta)$ shall denote the rotation about rotation axis $\boldsymbol{n}$ through angle θ. The axis direction is from the origin towards $\boldsymbol{n}$.

There are two conventions in use for specifying the angle of rotation. Either the angle is measured from the starting position of a point to its rotated position, or it is measured from its rotated position to its starting position. The first convention is the *position vector rotation* (PVR) convention, and the second convention is the *coordinate frame rotation* (CFR) convention. Figure 6.1 illustrates the two conventions for a point $\boldsymbol{r}$ that is rotated to a new position $\boldsymbol{r}'$ about an axis that is perpendicular to the plane of the figure. Thus, $\boldsymbol{r}' = \boldsymbol{R_n}(\theta_{\text{PVR}})(\boldsymbol{r}) = \boldsymbol{R_n}(-\theta_{\text{CFR}})(\boldsymbol{r})$. When an angle convention is not specified, the PVR convention is assumed.

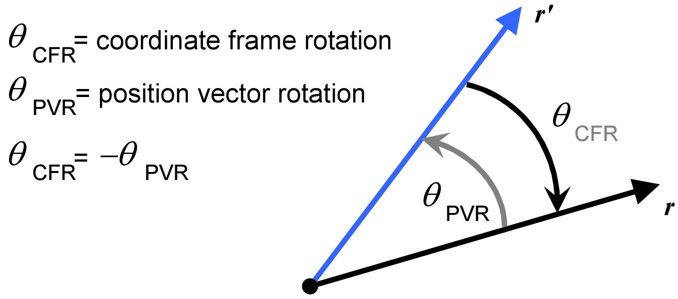

Figure 6.1 — Rotation between $\boldsymbol{r}$ and $\boldsymbol{r}'$ in two conventions

In the case of a change of coordinate basis operator $\boldsymbol{\Omega}_{F \leftarrow E}$, the direction cosine matrix $\boldsymbol{M}$ operating on coordinate three-tuples is length-preserving. Thus, Euler's rotation theorem associates a rotation operation with a change of coordinate basis operation. In particular, the associated rotation $\boldsymbol{R_n}(\theta_{\text{PVR}})$ rotates the basis vectors $\tilde{x}, \tilde{y}, \tilde{z}$ to coincide with corresponding basis vectors x, y, z. That is,

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{R_n}(\theta_{\text{PVR}})(\tilde{\boldsymbol{x}}) \\ \boldsymbol{y} &= \boldsymbol{R_n}(\theta_{\text{PVR}})(\tilde{\boldsymbol{y}}) \\ \boldsymbol{z} &= \boldsymbol{R_n}(\theta_{\text{PVR}})(\tilde{\boldsymbol{z}}) \end{aligned} \text{, or equivalently } \begin{aligned} \tilde{\boldsymbol{x}} &= \boldsymbol{R_n}(\theta_{\text{CFR}})(\boldsymbol{x}) \\ \tilde{\boldsymbol{y}} &= \boldsymbol{R_n}(\theta_{\text{CFR}})(\boldsymbol{y}) \\ \tilde{\boldsymbol{z}} &= \boldsymbol{R_n}(\theta_{\text{CFR}})(\boldsymbol{z}) \end{aligned}.$$

The two ways of viewing the vector space operation represented by the direction cosine matrix, either as a rotation or as a change of coordinate basis, are illustrated in Figure 6.2.

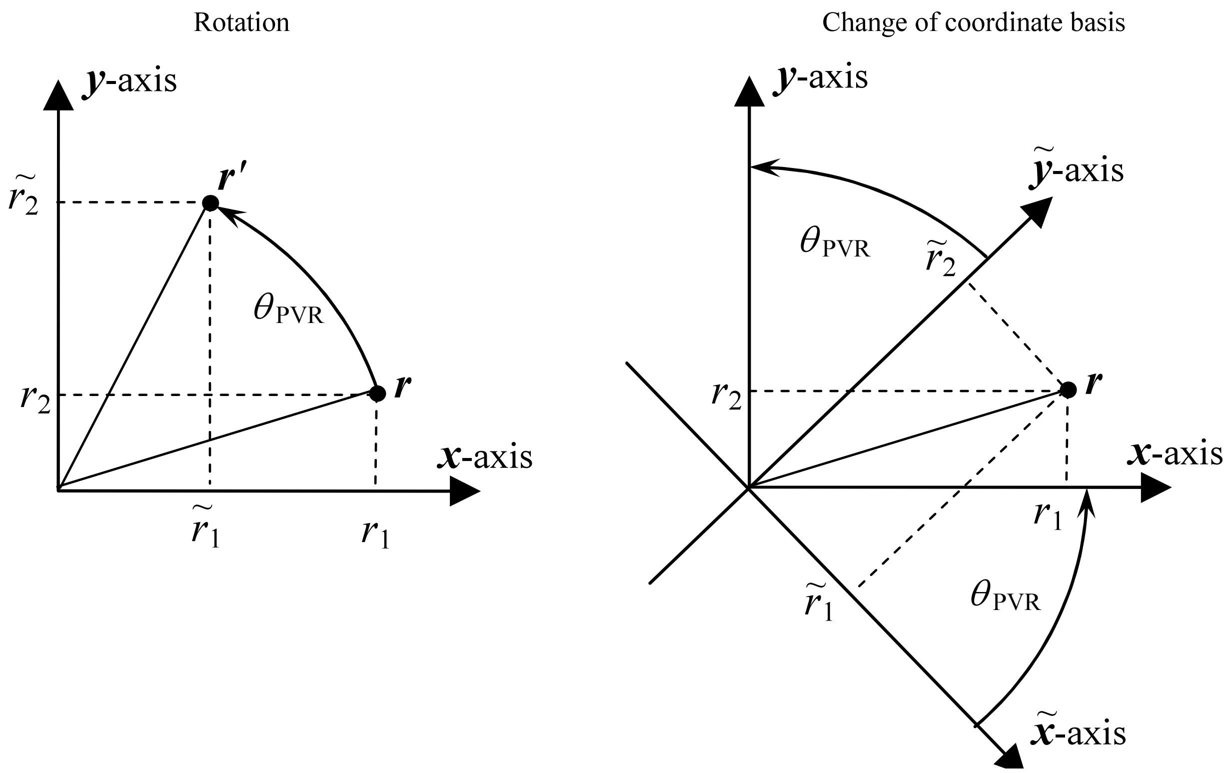

Figure 6.2 — Vector space operations

6.3 Consecutive operations

If $\boldsymbol{E}$, $\boldsymbol{F}$, $\boldsymbol{G}$ denotes three orthonormal frames with common origin, and if

$\boldsymbol{\Omega}_{F \leftarrow E}$ is the change of coordinate basis operator from $\boldsymbol{E}$ to $\boldsymbol{F}$, and

$\boldsymbol{\Omega}_{G \leftarrow F}$ is the change of coordinate basis operator from $\boldsymbol{F}$ to $\boldsymbol{G}$,

then the change of coordinate basis operator from $\boldsymbol{E}$ to $\boldsymbol{G}$ is given by the composition of operators in right-to-left operator order:

$$\boldsymbol{\Omega}_{G \leftarrow E} = \boldsymbol{\Omega}_{G \leftarrow F} \circ \boldsymbol{\Omega}_{F \leftarrow E} .$$

If an origin is designated for 3D Euclidean space, and $\boldsymbol{R_m}(\theta)$ and $\boldsymbol{R_n}(\varphi)$ are two rotation operators with respect to that origin, then the combined effect of the sequential combination of operation $\boldsymbol{R_m}(\theta)$ followed by operation $\boldsymbol{R_n}(\varphi)$ is ambiguous as stated. This is because the first operation $\boldsymbol{R_m}(\theta)$ will rotate the axis $\boldsymbol{n}$ to a new direction $\boldsymbol{n}' = \boldsymbol{R_m}(\theta)(\boldsymbol{n})$ and the sequential combination of these operations may be taken to mean that the first rotation is followed either by $\boldsymbol{R_n}(\varphi)$ or by $\boldsymbol{R_{n'}}(\varphi)$ and, in general, the final result will differ accordingly. Both choices are useful and important and need to be clearly distinguished. This International standard uses the term *space-fixed* for the un-rotated second axis case $\boldsymbol{n}$ and the term *body-fixed* for the rotated second axis case $\boldsymbol{n}'$. The possible results of the composition of two consecutive rotations in right-to-left operator order are:

$\boldsymbol{R_n}(\varphi) \circ \boldsymbol{R_m}(\theta)$ space-fixed composition, and

$\boldsymbol{R_{n'}}(\varphi) \circ \boldsymbol{R_m}(\theta)$ body-fixed composition.

In terms of coordinate computations, $\boldsymbol{n}$ and $\boldsymbol{m}$ have coordinates (n_1, n_2, n_3) and (m_1, m_2, m_3) in an orthonormal frame $\boldsymbol{S}$. The first rotation rotates the basis vectors to new directions resulting in a different basis. Denote the orthonormal frame of the second basis as $\boldsymbol{B}$. (It is useful to think of a copy of the basis vectors of $\boldsymbol{S}$ attached to a rigid body. The body with attached vectors are rotated by $\boldsymbol{R_m}(\theta)$ to form an orthonormal frame $\boldsymbol{B}$.) In the space-fixed case, $\boldsymbol{n}$ is interpreted as a coordinate $(n_1, n_2, n_3)_{\boldsymbol{S}}$ in $\boldsymbol{S}$. In the body-fixed case, $\boldsymbol{n}$ is interpreted as a coordinate $(n_1, n_2, n_3)_{\boldsymbol{B}}$ in $\boldsymbol{B}$. With respect to $\boldsymbol{S}$, $(n_1, n_2, n_3)_{\boldsymbol{B}}$ is a different vector. It is $\boldsymbol{n'}$, the result of rotating $\boldsymbol{n}$, and its coordinate representation in $\boldsymbol{S}$ may be computed by $\boldsymbol{n'} = (n'_1, n'_2, n'_3)_{\boldsymbol{S}} = \boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}}\left((n_1, n_2, n_3)_{\boldsymbol{B}}\right)$.

Change of coordinate basis operators may be utilized to compute the rotation operator $\boldsymbol{R_{n'}}(\varphi)$ in terms of $\boldsymbol{R_n}(\varphi)$:

$$\boldsymbol{R_{n'}}(\varphi) = \boldsymbol{\Omega}_{\boldsymbol{B}\leftarrow\boldsymbol{S}} \circ \boldsymbol{R_n}(\varphi) \circ \boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}} .$$

Since the operator $\boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}}^{-1} = \boldsymbol{\Omega}_{\boldsymbol{B}\leftarrow\boldsymbol{S}} = \boldsymbol{R_m}(\theta)$, the body-fixed case may be simplified:

$$\boldsymbol{R_{n'}}(\varphi) \circ \boldsymbol{R_m}(\theta) = \left(\boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}}^{-1} \circ \boldsymbol{R_n}(\varphi) \circ \boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}}\right) \circ \boldsymbol{R_m}(\theta) = \boldsymbol{R_m}(\theta) \circ \boldsymbol{R_n}(\varphi) \circ \left(\boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}} \circ \boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}}^{-1}\right) = \boldsymbol{R_m}(\theta) \circ \boldsymbol{R_n}(\varphi) .$$

Thus the two cases are simply expressed as:

$$\begin{aligned} &\boldsymbol{R_n}(\varphi) \circ \boldsymbol{R_m}(\theta) && \text{space-fixed composition, and} \\ &\boldsymbol{R_{n'}}(\varphi) \circ \boldsymbol{R_m}(\theta) = \boldsymbol{R_m}(\theta) \circ \boldsymbol{R_n}(\varphi) && \text{body-fixed composition.} \end{aligned} \tag{6.2}$$

NOTE Other terminology used for the space-fixed and body-fixed concepts include: extrinsic and intrinsic rotations; and fixed-frame and moving-frame.

6.4 Orientation specification

An *orientation specification* for an object orthonormal frame $\boldsymbol{E}$ with respect to a reference orthonormal frame $\boldsymbol{F}$ shall be specified by either:

a) The change of coordinate basis operator $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$ that converts a coordinate in orthonormal frame $\boldsymbol{E}$ to a corresponding coordinate in reference orthonormal frame $\boldsymbol{F}$, or

b) The rotation operator $\boldsymbol{R_m}(\theta)$ that would rotate the reference orthonormal frame $\boldsymbol{F}$ to align with the object orthonormal frame $\boldsymbol{E}$.

The rotation (b) that specifies the orientation of orthonormal frame $\boldsymbol{E}$ with respect to reference orthonormal frame $\boldsymbol{F}$ shall be denoted by $\boldsymbol{R}_{\boldsymbol{F}\to\boldsymbol{E}}$. The direction cosine matrices corresponding to (a) and (b) are the same. In that sense, the two ways of specifying an orientation are equivalent.

The operator in an orientation specification may be represented in any one of the forms delineated in 6.7.

Rotation operations (in a given rotation convention) and orientation specifications are closely related, but the relationship is not one-to-one. The rotations $\boldsymbol{R_m}(\theta + 2\pi k)$, where k is any positive or negative integer value, are distinct rotations that all correspond to the same orientation specification. Thus only the angle of rotation modulo 2π determines orientation. Large rotations (greater than one full revolution) are important in some applications, however, in this International Standard angles shall be considered equivalent modulo 2π.

6.5 Change of orientation reference frame

Given the specification of the orientation of an object orthonormal frame $\boldsymbol{E}$ with respect to one reference orthonormal frame $\boldsymbol{F}$, the orientation with respect to a second reference orthonormal frame $\boldsymbol{G}$ may be calculated directly if the orientation of the first reference frame with respect to the second is known.

In terms of change of coordinate basis orientation specifications:

$\boldsymbol{\Omega}_{F \leftarrow E}$ denotes the orientation of $\boldsymbol{E}$ with respect to $\boldsymbol{F}$,
$\boldsymbol{\Omega}_{G \leftarrow F}$ denotes the orientation of $\boldsymbol{F}$ with respect to $\boldsymbol{G}$, and
$\boldsymbol{\Omega}_{G \leftarrow E}$ denotes the orientation of $\boldsymbol{E}$ with respect to $\boldsymbol{G}$.

$\boldsymbol{\Omega}_{G \leftarrow E}$ may be computed by: $\boldsymbol{\Omega}_{G \leftarrow E} = \boldsymbol{\Omega}_{G \leftarrow F} \circ \boldsymbol{\Omega}_{F \leftarrow E}$.

In terms of rotation orientation specifications:

$\boldsymbol{R}_{F \to E}$ denotes the orientation of $\boldsymbol{E}$ with respect to $\boldsymbol{F}$,
$\boldsymbol{R}_{G \to F}$ denotes the orientation of $\boldsymbol{F}$ with respect to $\boldsymbol{G}$, and
$\boldsymbol{R}_{G \to E}$ denotes the orientation of $\boldsymbol{E}$ with respect to $\boldsymbol{G}$.

$\boldsymbol{R}_{G \to E}$ may be computed by: $\boldsymbol{R}_{G \to E} = \boldsymbol{R}_{G \to F} \circ \boldsymbol{R}_{F \to E}$.

6.6 Rodrigues' rotation formula

The notion of a rotation about an axis through a given rotation angle is independent of any selection of a Euclidean coordinate system (*i.e.*, coordinate free). If a rotation operator $\boldsymbol{R}_{\boldsymbol{n}}(\theta)$ rotates a point $\boldsymbol{r}$, the resulting rotated point $\boldsymbol{r}'$ may be computed using (coordinate free) vector space operations using Rodrigues' rotation formula (see [BERN]):

$$\boldsymbol{r}' = \cos(\theta)\boldsymbol{r} + (1 - \cos(\theta))(\boldsymbol{r} \bullet \boldsymbol{n})\boldsymbol{n} + \sin(\theta)\boldsymbol{n} \times \boldsymbol{r} \tag{6.3}$$

The terms may be rearranged to the alternate form:

$$\boldsymbol{r}' = \boldsymbol{r} + (1 - \cos(\theta))\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{r}) + \sin(\theta)\boldsymbol{n} \times \boldsymbol{r} \tag{6.4}$$

This formulation also applies to both PVR and CFR conventions.

6.7 Representations of Rotations

6.7.1 Representation degrees of freedom and computational complexity

A consequence of Euler's rotation theorem is that any rotation operation on 3D Euclidean space has three degrees of freedom and may be specified by three scalar numbers. That is explicitly the case with Euler angle conventions (see 6.7.4).

Other less compact specifications using four or more scalar parameters together with constraint rules are commonly used because they are more amenable to some computations, such as performing a rotation operation on a point, composing rotations, interpolating rotations, and other operations, and/or because these parameters can be measured or modelled directly. The Matrix representation (see 6.7.2) and the Quaternion representation (see 6.7.5) are in common use because the rotation of a point and the composition of rotations are directly computable as matrix or quaternion multiplications. Computing the composition of rotations in the Axis-angle representation (see 6.7.3) or in an Euler angle convention (see 6.7.4) usually require conversion to and from Matrix or Quaternion forms. All rotation representations defined in the remainder of this clause tacitly require an orthonormal basis for the coordinate representation of vectors.

The various representation methods in prevalent use present different tradeoffs with respect to storage size, computational complexity, speed, and error control. Thus the best representation is dependent on the requirements and computational environment of a user application. For this reason, different representations are in use and interoperability becomes an issue. This issue is compounded by the non-standard meaning of terms in prevalent use. To support interoperability and SRM operations, this International Standard defines these terms and identifies several representation methods as well as algorithms for key operations on and inter-conversions between the representation methods.

6.7.2 Matrix representation

A 3x3 matrix $\boldsymbol{M}$ is a rotation matrix, if it satisfies these properties:

$$\begin{aligned} &\det(\boldsymbol{M}) = 1 \\ &\boldsymbol{M}^{-1} = \boldsymbol{M}^{\mathrm{T}} \end{aligned} \tag{6.5}$$

Matrices satisfying these properties form an algebraic group with respect to matrix multiplication. This group is known as the *special orthogonal group* of degree 3, SO(3). In particular, the product of any two rotation matrices is itself a rotation matrix.

The operation of left matrix multiplication by $\boldsymbol{M}$ corresponds to a rotation by angle θ_{PVR} about the rotation axis spanned by the unit vector $\boldsymbol{n}$. The points that lie on the rotation axis are fixed points under the operation. The parameters $\boldsymbol{n}$ and θ_{PVR} are algorithmically determined as follows:

$$\text{If } \boldsymbol{M} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \theta_{\mathrm{PVR}} = \arccos\left(\left(\frac{\mathrm{Trace}(\boldsymbol{M}) - 1}{2}\right)\right) = \arccos\left(\left(\frac{(a_{11} + a_{22} + a_{33}) - 1}{2}\right)\right), \quad 0 \le \theta_{\mathrm{PVR}} \le \pi.$$

There are three cases for the computation of $\boldsymbol{n}$ that depend on the value of θ_{PVR}.

Case $\theta_{\mathrm{PVR}} = 0$: There is no rotation so $\boldsymbol{n}$ is indeterminate.

Case $0 < \theta_{\mathrm{PVR}} < \pi$: Let $\boldsymbol{n} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$, where:

$$\boldsymbol{v} = \begin{pmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{pmatrix}. \text{ In this case, } \|\boldsymbol{v}\| = 2\left|\sin(\theta_{\mathrm{PVR}})\right|.$$

Case: $\theta_{\mathrm{PVR}} = \pi$: First find the maximum diagonal element a_{11}, a_{22}, or a_{33} of $\boldsymbol{M}$. Then:

Sub-case: a_{11} is the maximum and $\boldsymbol{v} = (a_{11} + 1, a_{12}, a_{13})$.

Sub-case: a_{22} is the maximum and $\boldsymbol{v} = (a_{21}, a_{22} + 1, a_{23})$.

Sub-case: a_{33} is the maximum and $\boldsymbol{v} = (a_{31}, a_{32}, a_{33} + 1)$.

Finally $\boldsymbol{n} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$.

In all cases, $-\boldsymbol{n}$ and θ_{CFR} is also a solution.

NOTE 1 Matrix multiplication is generally not commutative.

NOTE 2 The matrix has nine parameters; however the constraints on the determinant and the transpose reduce the degrees of freedom to three.

A special case of a rotation matrix arises from a change of coordinate basis operation. If $\boldsymbol{E}$ and $\boldsymbol{F}$ are two orthonormal frames with common origin and respective bases $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, and $\boldsymbol{f}_1$, $\boldsymbol{f}_2$, $\boldsymbol{f}_3$, the matrix $\boldsymbol{M}$ corresponding to the coordinate basis operation $\boldsymbol{\Omega}_{F \leftarrow E}$ by the direction cosine matrix (see 6.2):

$$\boldsymbol{M} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_1 \bullet \boldsymbol{f}_1 & \boldsymbol{e}_2 \bullet \boldsymbol{f}_1 & \boldsymbol{e}_3 \bullet \boldsymbol{f}_1 \\ \boldsymbol{e}_1 \bullet \boldsymbol{f}_2 & \boldsymbol{e}_2 \bullet \boldsymbol{f}_2 & \boldsymbol{e}_3 \bullet \boldsymbol{f}_2 \\ \boldsymbol{e}_1 \bullet \boldsymbol{f}_3 & \boldsymbol{e}_2 \bullet \boldsymbol{f}_3 & \boldsymbol{e}_3 \bullet \boldsymbol{f}_3 \end{pmatrix} \tag{6.6}$$

$\boldsymbol{M}$ is also the matrix representation of the rotation specification $\boldsymbol{R}_{F \to E}$ of the orientation of orthonormal frame $\boldsymbol{E}$ with respect to reference frame $\boldsymbol{F}$.

6.7.3 Axis-angle representation

The *axis-angle* representation $(\boldsymbol{n}, \theta)$, for a given orthonormal frame, is a representation of a rotation $\boldsymbol{R}_{\boldsymbol{n}}(\theta)$ consisting of a unit vector $\boldsymbol{n} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^{\text{T}}$ and a rotation angle θ. This representation uses four scalar parameters n_1, n_2, n_3 and θ. The unit vector constraint $\|\boldsymbol{n}\| = 1$ reduces the degrees of freedom to three. The axis-angle representation is not unique. In particular, the axis-angle pairs $(\boldsymbol{n}, \theta)$ and $(-\boldsymbol{n}, -\theta)$ represent the same rotation, and $(\boldsymbol{n}, \theta_{\text{PVR}}) = (-\boldsymbol{n}, \theta_{\text{CFR}})$. When $\theta = 0$, $\boldsymbol{n}$ may be any unit vector or the zero vector.

NOTE A three parameter version in the form $(a_1, a_2, a_3) = (\theta n_1, \theta n_2, \theta n_3) = \theta \boldsymbol{n}$ is also in use. In this form, θ is non-negative and is computed as $\theta = \|(a_1, a_2, a_3)\|$ and $\boldsymbol{n} = \frac{1}{\theta}(a_1, a_2, a_3)$ when $\theta \neq 0$.

The operation of an axis-angle rotation $(\boldsymbol{n}, \theta)$ on 3D Euclidean space is given by Rodrigues' rotation formula (Equation (6.3)). There is no direct computational formulation of the composition of two axis-angle rotations in axis-angle form.

6.7.4 Principal rotations and Euler angle conventions

6.7.4.1 Principal rotations

Principal rotations are defined with respect to a given orthonormal frame. Unit axis vectors are represented in that basis by the coordinate 3-tuples: $\boldsymbol{x} = (1, 0, 0)$, $\boldsymbol{y} = (0, 1, 0)$, and $\boldsymbol{z} = (0, 0, 1)$. As an axis of rotation, each of these unit vectors is termed a *principal axis* of rotation. A rotation about a principal axis is termed a *principal rotation*. Some authors refer to these rotations as *elementary rotations*. The vector space operators: $\boldsymbol{R}_{\boldsymbol{x}}(\alpha)$, $\boldsymbol{R}_{\boldsymbol{y}}(\beta)$, and $\boldsymbol{R}_z(\gamma)$ denote the three principal rotations through the respective angles α, β, and γ modulo 2π.

6.7.4.2 Euler angles

Euler angles are a specification of a rotation obtained by the body-fixed composition of three consecutive principal rotations. There are twelve distinct ways to select a sequence of three principal axes and apply the principal rotations (24 if left-handed axes are considered)[19]. Each such ordered selection of axes is an *Euler angle convention*. There is little agreement among authors on names or notations for these conventions. There are numerous Euler angle conventions in use and many are named inconsistently. Some authors use a left-handed coordinate system. All coordinate systems in this International Standard are right-handed.

This International Standard adopts the following convention and notation for Euler angles: Given a 3-tuple of Euler angles (α,β,γ) the Euler convention specification shall be specified by a character string denoting the sequence of principal axes in the form A_1– A_2 – A_3 where each symbol A_1, A_2 , A_3 is one of the axis letters $\boldsymbol{x}$, $\boldsymbol{y}$, or z. Thus (α,β,γ) in the z-$\boldsymbol{x}$-z Euler convention is the composite of a principal rotation by angle α about the z-axis first, by angle β about $\boldsymbol{x}'$, the once rotated $\boldsymbol{x}$-axis, second, and by angle γ about z'', the twice rotated z-axis, for the third rotation. The resulting body-fixed composite is $\boldsymbol{R}_{z''}(\gamma)\circ\boldsymbol{R}_{\boldsymbol{x}'}(\beta)\circ\boldsymbol{R}_{z}(\alpha)$. Using Equation (6.2), the same result in un-rotated axes is $\boldsymbol{R}_{z}(\alpha)\circ\boldsymbol{R}_{\boldsymbol{x}}(\beta)\circ\boldsymbol{R}_{z}(\gamma)$. In general, the equivalent expressions in rotated and non-rotated principal axes is:

$$\boldsymbol{R}_{\mathrm{A}''_3}(\gamma)\circ\boldsymbol{R}_{\mathrm{A}'_2}(\beta)\circ\boldsymbol{R}_{\mathrm{A}_1}(\alpha)=\boldsymbol{R}_{\mathrm{A}_1}(\alpha)\circ\boldsymbol{R}_{\mathrm{A}_2}(\beta)\circ\boldsymbol{R}_{\mathrm{A}_3}(\gamma)$$

where the second and third axes sequentially are rotated:

$$\begin{aligned}\mathrm{A}'_2&=\boldsymbol{R}_{\mathrm{A}_1}(\alpha)\left(\mathrm{A}_2\right),\text{ and}\\ \mathrm{A}''_3&=\boldsymbol{R}_{\mathrm{A}'_2}(\beta)\circ\boldsymbol{R}_{\mathrm{A}_1}(\alpha)\left(\mathrm{A}_3\right).\end{aligned}\tag{6.7}$$

The three angles representing a rotation in a given Euler angle convention are not necessarily unique modulo 2π. The conditions that result in non-unique angle 3-tuples are given in Table 6.4 for the z-$\boldsymbol{x}$-z Euler angle convention and in Table 6.7 for the $\boldsymbol{x}$-$\boldsymbol{y}$-z and z-$\boldsymbol{y}$-$\mathbf{x}$ Euler angle conventions (see also 6.7.4.5).

EXAMPLE Substituting in Equation (6.7), the Euler sequence (ψ,θ,φ) in the Euler z-$\boldsymbol{y}$-$\boldsymbol{x}$ convention is $\boldsymbol{R}_{x''}(\varphi)\circ\boldsymbol{R}_{y'}(\theta)\circ\boldsymbol{R}_{z}(\psi)$ or equivalently $\boldsymbol{R}_{z}(\psi)\circ\boldsymbol{R}_{y}(\theta)\circ\boldsymbol{R}_{x}(\varphi)$.

There are no direct computational formulations for the operation of an Euler angle rotation on 3D Euclidean space or for representing the composition of two Euler angle rotations as a single Euler angle rotation. For these computations, the principal rotation sequence is commonly realized as a product of matrices or quaternions.

NOTE Some authors denote Euler conventions that use distinct (non-repeating) axes as Tait-Bryan conventions.

[19] There cannot be two consecutive rotations on the same axis as they would combine to a single rotation. Thus, among right-handed axis systems, there are 3 choices for the first rotation axis, 2 choices each for the second and third rotation axes to avoid repeating the preceding axis choice (3x2x2=12).

6.7.4.3 The z-x-z convention

In the z-x-z Euler convention, the initial xy-plane and the final rotated $x''y''$-plane generally intersect in a line. This line is termed the *line of nodes* for this convention. The Euler angles in the z-x-z convention are the three angles defined as follows:

α is the angle between the line of nodes and the x''-axis,
β is the angle between the z-axis and the z''-axis, and
γ is the angle between the x-axis and the line of nodes.

In the case that the initial xy-plane lies in the final rotated $x''y''$-plane, $\beta = 0$ or $\beta = \pi$ (see 6.7.4.5).

In some contexts α, β, γ are known, respectively, as the *spin* angle, the *nutation* angle, and the *precession* angle. These three angles specify the principal rotation angles the body-fixed composition of the z-axis principal rotation followed by (the rotated) x'-axis principal rotation followed by the (twice rotated) z''-axis principal rotation. The sequence of body-fixed rotations is illustrated in Figure 6.2. The resulting composite rotation is $\boldsymbol{R}_{z''}(\gamma) \circ \boldsymbol{R}_{x'}(\beta) \circ \boldsymbol{R}_{z}(\alpha) = \boldsymbol{R}_{z}(\alpha) \circ \boldsymbol{R}_{x}(\beta) \circ \boldsymbol{R}_{z}(\gamma)$.

6.7.4.4 Tait-Bryan angles

Euler angle conventions that use all three principal axes are sometimes referred to as *Tait-Bryan angles*. In particular, the angles in the x-y-z and z-y-x Euler conventions are variously termed Tait-Bryan angles, *Cardano angles*, or *nautical angles*. The various names given to these angle symbols include:

φ roll or bank or tilt,
θ pitch or elevation, and
ψ yaw or heading or azimuth (see Figure 6.4).

In the x-y-z Euler convention the line of nodes is the intersection of the xy-plane and the final rotated $y''z''$-plane. The Euler angles in this convention are defined as follows:

φ is the angle between the line of nodes and the y''-axis,
θ is the angle between x''-axis and the xy-plane, (equivalently, the z-axis and the $y''z''$-plane), and
ψ is the angle between the y-axis and the line of nodes.

These three angles (φ, θ, ψ) specify the following body-fixed composition of consecutive principal rotations:

$$\boldsymbol{R}_{z''}(\psi) \circ \boldsymbol{R}_{y'}(\theta) \circ \boldsymbol{R}_{x}(\varphi) = \boldsymbol{R}_{x}(\varphi) \circ \boldsymbol{R}_{y}(\theta) \circ \boldsymbol{R}_{z}(\psi) \qquad x\text{-}y\text{-}z \text{ Euler convention.}$$

In the z-y-x Euler convention the line of nodes is the intersection of the yz-plane and the final rotated $x''y''$-plane. The Euler angles in this convention are defined as follows:

φ is the angle between the line of nodes and the y-axis,
θ is the angle between x-axis and the $x''y''$-plane, (equivalently, the z''-axis and the yz-plane), and
ψ is the angle between the y''-axis and the line of nodes.

These three angles (ψ, θ, φ) specify the following body-fixed composition of consecutive principal rotations:

$$\boldsymbol{R}_{x''}(\varphi) \circ \boldsymbol{R}_{y'}(\theta) \circ \boldsymbol{R}_{z}(\psi) = \boldsymbol{R}_{z}(\psi) \circ \boldsymbol{R}_{y}(\theta) \circ \boldsymbol{R}_{x}(\varphi) \qquad z\text{-}y\text{-}x \text{ Euler convention.}$$

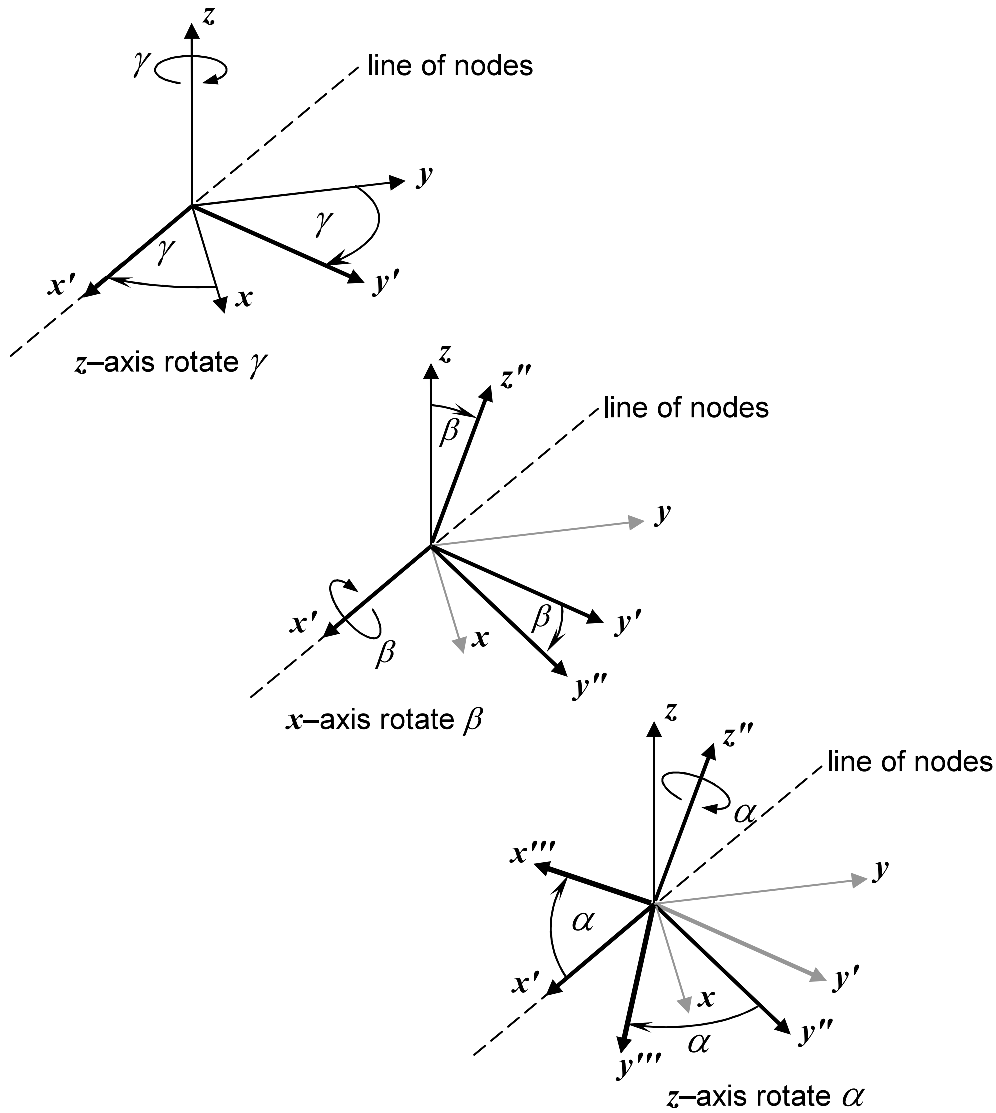

Figure 6.3 — Euler *z*-*x*-*z* body-fixed realization

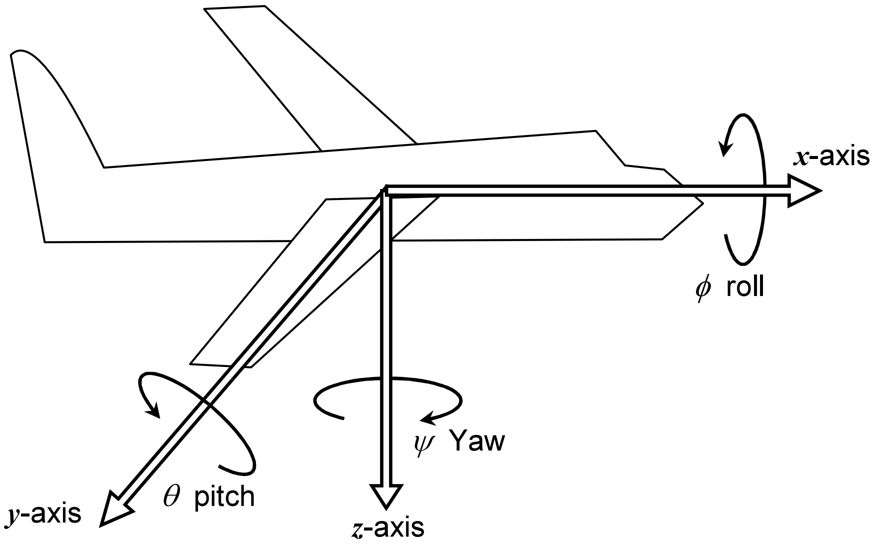

Figure 6.4 — Tait-Bryan angles

6.7.4.5 Gimbal lock

The term *gimbal lock* refers to a gyroscope mounted in three nested gimbals to provide three degrees of rotational freedom. Each mounting scheme corresponds to an Euler angle convention. In any such mounting scheme, there exist critical angles for the middle gimbal that reduce the rotational degrees of freedom from three to two. In those critical configurations, the gimbals lie in a single plane and rotation within that plane is figuratively "locked out" by the gimbal mechanism. This loss of a degree of freedom is termed "gimbal lock".

In the case of the Euler angle z-x-z rotation convention, it is assumed that the xy-plane and $x''y''$-plane intersect in a line (the line on nodes). That assumption is met when (modulo 2π) $\beta \neq 0$ and $\beta \neq \pi$. If not, $\beta = 0$ or $\beta = \pi$ and the consecutive rotations collapse down to a single principal rotation:

$$\begin{aligned} \beta = 0: &\quad \boldsymbol{R}_z(\alpha) \circ \boldsymbol{R}_x(0) \circ \boldsymbol{R}_z(\gamma) = \boldsymbol{R}_z(\alpha) \circ \boldsymbol{R}_z(\ \gamma) = \boldsymbol{R}_z(\alpha + \gamma) \\ \beta = \pi: &\quad \boldsymbol{R}_z(\alpha) \circ \boldsymbol{R}_x(\pi) \circ \boldsymbol{R}_z(\gamma) = \boldsymbol{R}_z(\alpha) \circ \boldsymbol{R}_z(-\gamma) \circ \boldsymbol{R}_x(\pi) = \boldsymbol{R}_z(\alpha - \gamma) \circ \boldsymbol{R}_x(\pi) \end{aligned} \tag{6.8}$$

NOTE 1 This situation is illustrated by a spinning table top. The top spins on its spin-axis and precesses about the precession-axis. The angle between the spin- and precession-axes is the nutation angle. When the spin-axis is perfectly vertical (either upright or upside down), the nutation angle is 0 or π and the spin- and precession-axes become indistinguishable from each other as indicated in Equation (6.8).

In the case of the Euler angle x-y-z convention (Tait-Bryan) it is assumed that the xy-plane and $\tilde{y}\tilde{z}$-plane intersect in a line (the line of nodes). That assumption is met when $\theta \neq \pm\pi/2$ modulo 2π. If not, $\theta = \pm\pi/2$ and the $\tilde{x}$-axis becomes parallel to the z-axis and the consecutive rotations collapse down to a single principal rotation:

$$\begin{aligned} \theta = +\pi/2: &\quad \boldsymbol{R}_x(\varphi) \circ \boldsymbol{R}_y(+\pi/2) \circ \boldsymbol{R}_z(\psi) = \boldsymbol{R}_x(\psi + \varphi) \circ \boldsymbol{R}_y(+\pi/2) \\ \theta = -\pi/2: &\quad \boldsymbol{R}_x(\psi) \circ \boldsymbol{R}_y(-\pi/2) \circ \boldsymbol{R}_z(\varphi) = \boldsymbol{R}_x(\psi - \varphi) \circ \boldsymbol{R}_y(-\pi/2) \end{aligned} \tag{6.9}$$

The case of the Euler angle z-y-x convention has a similar result:

$$\begin{aligned} \theta = +\pi/2: \quad & \boldsymbol{R}_z(\psi) \circ \boldsymbol{R}_y(+\pi/2) \circ \boldsymbol{R}_x(\varphi) = \boldsymbol{R}_z(\psi + \varphi) \circ \boldsymbol{R}_y(+\pi/2) \\ \theta = -\pi/2: \quad & \boldsymbol{R}_z(\psi) \circ \boldsymbol{R}_y(-\pi/2) \circ \boldsymbol{R}_x(\varphi) = \boldsymbol{R}_z(\psi - \varphi) \circ \boldsymbol{R}_y(-\pi/2) \end{aligned} \tag{6.10}$$

NOTE 2 This situation is illustrated by an aircraft as in Figure 6.4. When the aircraft either climbs vertically, or dives vertically, roll-rotation cannot be distinguished from (plus or minus) yaw-rotation. This occurs at critical pitch angles of $\theta = \pm\pi/2$ as indicated in Equation (6.9).

6.7.5 Quaternion representation

6.7.5.1 Quaternion notations and conventions

The quaternion system is a 4-dimensional vector space together with a vector multiplication operation that forms a non-commutative associative algebra. In analogy to complex numbers that are written as $a + \boldsymbol{i}b,\ \boldsymbol{i}^2 = -1$, quaternion axes $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, are defined with the following relationships: $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. There are several notational conventions in use including the three termed in this International Standard as the *Hamilton form*, the *4-tuple form*, and the *scalar vector form*. In these notation forms a quaternion $\boldsymbol{q}$ is denoted as follows:

$\boldsymbol{q} = e_0 + e_1\boldsymbol{i} + e_2\boldsymbol{j} + e_3\boldsymbol{k}$	Hamilton form
$\boldsymbol{q} = (e_0, e_1, e_2, e_3)$	4-tuple form
$\boldsymbol{q} = (e_0, \boldsymbol{e}),\ \boldsymbol{e} = (e_1 \quad e_2 \quad e_3)^{\mathrm{T}}$	scalar vector form

where e_0, e_1, e_2, e_3 are scalar values.

The e_0 value is termed the *real* (or “scalar”) part of $\boldsymbol{q}$ and (e_1, e_2, e_3) is termed the *imaginary* (or “vector”) part of $\boldsymbol{q}$. The remainder of this clause uses the scalar vector form.

NOTE 1 In the literature, the component order of the scalar vector form is sometimes reversed: $\boldsymbol{q} = (\boldsymbol{e}, e_0)$.

NOTE 2 A unit quaternion (see below) in 4-tuple form is also termed the *Euler parameters* (or the *Euler-Rodrigues parameters*) of a rotation. In the literature, the real part of the 4-tuple form is sometimes placed last: $\boldsymbol{q} = (e_1, e_2, e_3, e_4)$ where $e_4 = e_0$.

6.7.5.2 Quaternion algebra

Quaternion multiplication and other operations are defined in Annex A in all the three notational forms. Given quaternions $\boldsymbol{q} = (e_0, \boldsymbol{e})$ and $\boldsymbol{p} = (d_0, \boldsymbol{d})$, A.10 defines:

the product $\boldsymbol{pq} = \left(\left(d_0 e_0 - \boldsymbol{d} \bullet \boldsymbol{e} \right), \left(e_0 \boldsymbol{d} + d_0 \boldsymbol{e} + \boldsymbol{d} \times \boldsymbol{e} \right) \right)$,

the conjugate $\boldsymbol{q}^* = (e_0, -\boldsymbol{e})$,

the modulus $|\boldsymbol{q}| = \sqrt{\boldsymbol{qq}^*} = \sqrt{e_0^2 + e_1^2 + e_2^2 + e_3^2}$,

where $\boldsymbol{qq}^* = \boldsymbol{q}^*\boldsymbol{q} = \left(e_0^2 + e_1^2 + e_2^2 + e_3^2,\ \boldsymbol{0} \right)$.

A quaternion $\boldsymbol{q}$ is a *unit quaternion* if $|\boldsymbol{q}| = 1$. In that case $\boldsymbol{q}\boldsymbol{q}^* = \boldsymbol{q}^*\boldsymbol{q} = (1, \mathbf{0})$ which is the multiplicative identity so that, for a unit quaternion, its conjugate is its multiplicative inverse $\boldsymbol{q}^{-1} = \boldsymbol{q}^*$. Any unit quaternion may be expressed in the form:

$$\boldsymbol{q} = (\cos(\theta/2), \sin(\theta/2)\boldsymbol{n}) \tag{6.11}$$

where:

$\boldsymbol{n} = \dfrac{1}{\|\boldsymbol{e}\|}\boldsymbol{e}$ is a unit vector in 3D space,

$\theta = 2 \cdot \operatorname{arctan2}\left(\sqrt{e_1^2 + e_2^2 + e_3^2}, e_0\right)$.

NOTE The two argument arctangent function arctan2() is defined in Annex A.

6.7.5.3 Quaternion operators on 3D Euclidean space

Each quaternion $\boldsymbol{q}$ corresponds to a transformation of 3D Euclidean space as follows. If $\boldsymbol{r}$ is a vector in 3D Euclidean space, the corresponding quaternion is formed by using 0 for the real part and $\boldsymbol{r}$ for the imaginary part $(0, \boldsymbol{r})$. A unit quaternion $\boldsymbol{q}$ operates on $(0, \boldsymbol{r})$ by left multiplying with $\boldsymbol{q}$ and right multiplying with its conjugate $\boldsymbol{q}^*$. The real part of the product $\boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^* = (r_0', \boldsymbol{r}')$, is 0. Thus, $\boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^* = (0, \boldsymbol{r}')$ is pure imaginary and the quaternion $\boldsymbol{q}$ associates $\boldsymbol{r}'$ with $\boldsymbol{r}$. Symbolically the operation on $\boldsymbol{r}$ is:

$$\boldsymbol{r} \mapsto \boldsymbol{r}' = \mathit{imaginary\ part}\{\boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^*\}. \tag{6.12}$$

This is equivalent to:

$$\boldsymbol{r}' = \left(e_0^2 - \boldsymbol{e} \bullet \boldsymbol{e}\right)\boldsymbol{r} + 2(\boldsymbol{e} \bullet \boldsymbol{r})\boldsymbol{e} + 2e_0\boldsymbol{e} \times \boldsymbol{r}. \tag{6.13}$$

$-\boldsymbol{q} = (-e_0, -\boldsymbol{e})$ produces the same $\boldsymbol{r}'$ so that $\boldsymbol{q}$ and $-\boldsymbol{q}$ produce equivalent rotations.

If $\boldsymbol{q} = (\cos(\theta/2), \sin(\theta/2)\boldsymbol{n})$ is a unit quaternion, Equation (6.13) reduces to the Rodrigues rotation formula for a clockwise rotation about $\boldsymbol{n}$ through angle θ:

$$\boldsymbol{r}' = \cos(\theta)\boldsymbol{r} + (1 - \cos(\theta))(\boldsymbol{n} \bullet \boldsymbol{r})\boldsymbol{n} + \sin(\theta)\boldsymbol{n} \times \boldsymbol{r}.$$

A non-zero quaternion $\boldsymbol{p}$ and its corresponding unit quaternion $\boldsymbol{q} = \dfrac{\boldsymbol{p}}{|\boldsymbol{p}|}$ perform the same rotation

$$\boldsymbol{p}(0, \boldsymbol{r})\boldsymbol{p}^{-1} = \boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^*.$$

For this reason, some authors use $\boldsymbol{p}(0, \boldsymbol{r})\boldsymbol{p}^{-1}$ operations for any non-zero quaternion while others use the $\boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^*$ operator and restrict operations only to unit quaternions.

The quaternion representation of rotation facilitates the computation of the composition of two rotations.

If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are two unit quaternions, the composite rotation on $\boldsymbol{r}$ that is obtained by first rotating with the rotation induced by $\boldsymbol{q}_1$ and then rotating the result with the rotation induced by $\boldsymbol{q}_2$ is the same as the single rotation induced by the product $\boldsymbol{q}_2\boldsymbol{q}_1$ since $\boldsymbol{q}_2\{\boldsymbol{q}_1(0, \boldsymbol{r})\boldsymbol{q}_1^*\}\boldsymbol{q}_2^* = \boldsymbol{q}_2\boldsymbol{q}_1(0, \boldsymbol{r})\boldsymbol{q}_1^*\boldsymbol{q}_2^* = \{\boldsymbol{q}_2\boldsymbol{q}_1\}(0, \boldsymbol{r})\{\boldsymbol{q}_2\boldsymbol{q}_1\}^*$.

6.7.6 Representation summary

Some important attributes of the representations in this section are summarized in Table 6.1.

Table 6.1 — Summary of representation attributes

Representation type	Data compo-nents	Data constraints	Ambiguities (modulo 2π)	Composition	Inverse
Axis-angle $(\boldsymbol{n}, \theta)$	4	$\lVert\boldsymbol{n}\rVert = 1$	$(\boldsymbol{n}, \theta)$ is equivalent to $(-\boldsymbol{n}, -\theta)$. If $\theta = 0$, $\boldsymbol{n}$ is indeterminate	Convert to/from another representation for the operation	$(\boldsymbol{n}, -\theta)$ or $(-\boldsymbol{n}, \theta)$
Matrix $\boldsymbol{R}$	9	$\det(\boldsymbol{R}) = 1$ $\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}^{-1}$	None	Matrix multiplication	$\boldsymbol{R}^{\mathrm{T}}$
Euler angle conventions	3	None	2 or more z-x-z convention: see Table 6.4 Tait-Bryan z-y-x or x-y-z angles: see Table 6.5 and Table 6.6	Convert to/from another representation for the operation (see Note 2)	See Note 1
Unit quaternion $\boldsymbol{q}$	4	unit constraint: $\boldsymbol{q}\boldsymbol{q}^{*} = 1$	$\boldsymbol{q}$ is equivalent to $-\boldsymbol{q}$ (see Note 3)	Quaternion multiplication	$\boldsymbol{q}^{*}$ or $-\boldsymbol{q}^{*}$

NOTE 1 The inverse in the Euler angle z-x-z convention is

$$\left[\boldsymbol{R}_z(\alpha)\circ\boldsymbol{R}_x(\beta)\circ\boldsymbol{R}_z(\gamma)\right]^{-1} = \boldsymbol{R}_z(-\gamma)\circ\boldsymbol{R}_x(-\beta)\circ\boldsymbol{R}_z(-\alpha)$$

The inverse in the Euler angle x-y-z and z-y-x conventions (Tait-Bryan angles) are

$$\left[\boldsymbol{R}_x(\phi)\circ\boldsymbol{R}_y(\theta)\circ\boldsymbol{R}_z(\psi)\right]^{-1} = \boldsymbol{R}_z(-\psi)\circ\boldsymbol{R}_y(-\theta)\circ\boldsymbol{R}_x(-\phi)$$

$$\left[\boldsymbol{R}_z(\psi)\circ\boldsymbol{R}_y(\theta)\circ\boldsymbol{R}_x(\phi)\right]^{-1} = \boldsymbol{R}_x(-\phi)\circ\boldsymbol{R}_y(-\theta)\circ\boldsymbol{R}_z(-\psi)$$

NOTE 2 The composition of Euler angle operations may also be performed in a "direct" method that involves lengthy expressions combining forward and inverse trigonometric functions.

NOTE 3 Formulae such as Equation (6.13) require the unit quaternion constraint. Other useful relationships such as Equation (6.12) do not have that requirement. For that reason, some applications do not enforce the unit constraint. In the unconstrained case, every non-zero scalar multiple of a given quaternion is rotationally equivalent to it.

6.8 Inter-converting between rotations representations

6.8.1 Euler angle conventions and matrix representation

6.8.1.1 Matrix forms of principal rotations

For notation convenience, given a principal axis $\boldsymbol{a}$ ($\boldsymbol{a}=\boldsymbol{x}$ or $\boldsymbol{y}$ or $\boldsymbol{z}$), $\boldsymbol{R}_{\boldsymbol{a}}(\omega)$ shall denote a principal rotation with angle specification in the PVR convention and $\boldsymbol{\Omega}_{\boldsymbol{a}}(\omega)$ shall denote a principal rotation with angle specification in the CFR convention. In particular, $\boldsymbol{\Omega}_{\boldsymbol{a}}(\omega)=\boldsymbol{R}_{\boldsymbol{a}}(-\omega)$. The matrix representations of principal rotations in this notation are given in Table 6.2.

Table 6.2 — Principal rotations as matrix operators

Name	Notation	Matrix operator (left multiplication)
x-axis principal rotation CFR convention	$\boldsymbol{\Omega}_{\boldsymbol{x}}(\omega_1)$	$\boldsymbol{\Omega}_X(\omega_1)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\omega_1) & \sin(\omega_1)\\ 0 & -\sin(\omega_1) & \cos(\omega_1)\end{pmatrix}$, where ω_1 is the angle of rotation.
x-axis principal rotation PVR convention	$\boldsymbol{R}_{\boldsymbol{x}}(\omega_1)$	$\boldsymbol{R}_X(\omega_1)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\omega_1) & -\sin(\omega_1)\\ 0 & \sin(\omega_1) & \cos(\omega_1)\end{pmatrix}$, where ω_1 is the angle of rotation.
y-axis principal rotation CFR convention	$\boldsymbol{\Omega}_{\boldsymbol{y}}(\omega_2)$	$\boldsymbol{\Omega}_Y(\omega_2)=\begin{pmatrix}\cos(\omega_2) & 0 & -\sin(\omega_2)\\ 0 & 1 & 0\\ \sin(\omega_2) & 0 & \cos(\omega_2)\end{pmatrix}$, where ω_2 is the angle of rotation.
y-axis principal rotation PVR convention	$\boldsymbol{R}_{\boldsymbol{y}}(\omega_2)$	$\boldsymbol{R}_Y(\omega_2)=\begin{pmatrix}\cos(\omega_2) & 0 & \sin(\omega_2)\\ 0 & 1 & 0\\ -\sin(\omega_2) & 0 & \cos(\omega_2)\end{pmatrix}$, where ω_2 is the angle of rotation.
z-axis principal rotation CFR convention	$\boldsymbol{\Omega}_{\boldsymbol{z}}(\omega_3)$	$\boldsymbol{\Omega}_Z(\omega_3)=\begin{pmatrix}\cos(\omega_3) & \sin(\omega_3) & 0\\ -\sin(\omega_3) & \cos(\omega_3) & 0\\ 0 & 0 & 1\end{pmatrix}$, where ω_3 is the angle of rotation.
z-axis principal rotation PVR convention	$\boldsymbol{R}_{\boldsymbol{z}}(\omega_3)$	$\boldsymbol{R}_Z(\omega_3)=\begin{pmatrix}\cos(\omega_3) & -\sin(\omega_3) & 0\\ \sin(\omega_3) & \cos(\omega_3) & 0\\ 0 & 0 & 1\end{pmatrix}$, where ω_3 is the angle of rotation.

6.8.1.2 The z-x-z Euler angle convention

The angle sequence (α,β,γ) in the Euler z-x-z convention is converted to a matrix $\boldsymbol{M}$ by forming the matrix product of the corresponding three principal rotation matrices specified in Table 6.2. The resulting matrix is given in Equation (6.14).

$$\boldsymbol{M} = \boldsymbol{R}_{z''}(\gamma)\circ\boldsymbol{R}_{x'}(\beta)\circ\boldsymbol{R}_{z}(\alpha) = \boldsymbol{R}_{z}(\alpha)\circ\boldsymbol{R}_{x}(\beta)\circ\boldsymbol{R}_{z}(\gamma) =$$
$$\begin{pmatrix} \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & -\sin\gamma\cos\alpha - \cos\beta\cos\gamma\sin\alpha & \sin\beta\sin\alpha \\ \cos\beta\sin\gamma\cos\alpha + \cos\gamma\sin\alpha & \cos\beta\cos\alpha\cos\gamma - \sin\alpha\sin\gamma & -\sin\beta\cos\alpha \\ \sin\beta\sin\gamma & \sin\beta\cos\gamma & \cos\beta \end{pmatrix} \tag{6.14}$$

Conversely, given a matrix $\boldsymbol{M}$ with elements a_{ij}, the equation may solved for the principal rotation factors $\boldsymbol{R}_{z}(\alpha)\circ\boldsymbol{R}_{x}(\beta)\circ\boldsymbol{R}_{z}(\gamma)$, and therefore solved for angles (α,β,γ). The solution is given in Table 6.3.

Table 6.3 — Principal factors for the Euler z-x-z convention

Case	Principal factors for rotation $\boldsymbol{R}_{z''}(\gamma)\circ\boldsymbol{R}_{x'}(\beta)\circ\boldsymbol{R}_{z}(\alpha) = \boldsymbol{R}_{z}(\alpha)\circ\boldsymbol{R}_{x}(\beta)\circ\boldsymbol{R}_{z}(\gamma)$ (all angles modulo 2π, $\boldsymbol{M}=[\,a_{ij}\,]$)		
$a_{33} \neq \pm 1$	$\beta = \arccos(a_{33})$ [principal value] $0 < \beta < \pi$	$\gamma = \text{arctan2}(a_{31}, a_{32})$	$\alpha = \text{arctan2}(a_{13}, -a_{23})$
	$\beta = \arccos(a_{33})$ [2π − principal value] $\pi < \beta < 2\pi$	$\gamma = \text{arctan2}(-a_{31}, -a_{32})$	$\alpha = \text{arctan2}(-a_{13}, a_{23})$
$a_{33} = -1$	$\beta = \pi$	any value of γ	$\alpha = \text{arctan2}(a_{21}, a_{11}) + \gamma$
$a_{33} = +1$	$\beta = 0$	any value of γ	$\alpha = \text{arctan2}(a_{21}, a_{11}) - \gamma$

In the case $a_{33} \neq \pm 1$, arccos() is multi-valued so that there are two valid solution sets depending on the quadrants selected for arccosine values[20]. The principal value solution is the commonly used one. The two argument arctangent function arctan2() is defined in Annex A.

In the case $a_{33} = -1$, using trigonometric identities, the matrix expression reduces to :

$$\boldsymbol{R}_{z}(\alpha)\circ\boldsymbol{R}_{x}(\pi)\circ\boldsymbol{R}_{z}(\gamma) = \begin{pmatrix} \cos(\alpha-\gamma) & \sin(\alpha-\gamma) & 0 \\ \sin(\alpha-\gamma) & -\cos(\alpha-\gamma) & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

[20] Computer library functions such as acos() return the principal value only. The second solution for β may be obtained by subtracting the principal value from 2π.

For this reason, only the difference of the other two angles can be determined by using $\alpha - \gamma = \mathrm{arctan2}(a_{21}, a_{11})$. Therefore, all values are valid for α if $\gamma = \mathrm{arctan2}(a_{21}, a_{11}) + \alpha$. The case $a_{31} = +1$ is similar to the previous case with the sum of the angles determined by using $\gamma + \alpha = \mathrm{arctan2}(a_{21}, a_{11})$. These two cases correspond to the gimbal lock Equation (6.8).

As seen in the preceding tables, the three angle sequence corresponding to a given rotation or orientation operator is not unique modulo 2π. Two sequences, $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ of *z-x-z* principal factors specify the same operator if they satisfy one the criteria specified in Table 6.4.

Table 6.4 — Equivalence of *z-x-z* principal factor sequences

Case (equality modulo 2π)	**Criteria for the equivalence of angle sequences $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ for principal factor *z-x-z* sequences**
$\beta_1 = \beta_2$	$\alpha_1 = \alpha_2,\ \gamma_1 = \gamma_2\ [\beta_1, \beta_2 \neq 0 \text{ or } \pi]$ (in)equalities modulo 2π
$\lvert\beta_1 + \beta_2\rvert = 2\pi$	$\lvert\alpha_2 - \alpha_1\rvert = \pi,\ \lvert\gamma_2 - \gamma_1\rvert = \pi\ [\beta_1, \beta_2 \neq 0 \text{ or } \pi]$ (in)equalities modulo 2π
$\beta_1 = \beta_2 = \pi$	$\alpha_1 - \gamma_1 = \alpha_2 - \gamma_2$ equality modulo 2π
$\beta_1 = \beta_2 = 0$	$\alpha_1 + \gamma_1 = \alpha_2 + \gamma_2$ equality modulo 2π

6.8.1.3 The Tait-Bryan conventions

The Euler angle sequences (φ, θ, ψ) in convention *x-y-z* and (ψ, θ, φ) in convention *z-y-x* are converted to respective matrices $\boldsymbol{M}$ by forming the matrix product of the corresponding three principal rotation matrices specified in Table 6.2. The resulting matrices are given in Equation (6.15).

Conversely, given matrix $\boldsymbol{M}$ with elements a_{ij}, the equation may solved for the principal rotation factors $\boldsymbol{R}_z(\psi) \circ \boldsymbol{R}_y(\theta) \circ \boldsymbol{R}_x(\varphi)$, and therefore solved for angles (φ, θ, ψ). The solution is given in Table 6.5.

$$\boldsymbol{M} = \boldsymbol{R}_{z''}(\psi) \circ \boldsymbol{R}_{y'}(\theta) \circ \boldsymbol{R}_x(\varphi) = \boldsymbol{R}_x(\varphi) \circ \boldsymbol{R}_y(\theta) \circ \boldsymbol{R}_z(\psi) =$$
$$\begin{pmatrix} \cos\psi\cos\theta & -\sin\psi\cos\theta & \sin\theta \\ \cos\psi\sin\theta\sin\varphi + \sin\psi\cos\varphi & -\sin\psi\sin\theta\sin\varphi + \cos\psi\cos\varphi & -\cos\theta\sin\varphi \\ -\cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi & \sin\psi\sin\theta\cos\varphi + \cos\psi\sin\varphi & \cos\theta\cos\varphi \end{pmatrix}$$

$$\boldsymbol{M} = \boldsymbol{R}_{x''}(\varphi) \circ \boldsymbol{R}_{y'}(\theta) \circ \boldsymbol{R}_z(\psi) = \boldsymbol{R}_z(\psi) \circ \boldsymbol{R}_y(\theta) \circ \boldsymbol{R}_x(\varphi) =$$
$$\begin{pmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\varphi - \sin\psi\cos\varphi & \cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\varphi + \cos\psi\cos\varphi & \sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi \\ -\sin\theta & \cos\theta\sin\varphi & \cos\theta\cos\varphi \end{pmatrix} \tag{6.15}$$

Conversely, given matrix $\boldsymbol{M}$ with elements a_{ij}, the equation may solved for the angle sequence of the principal rotation factors. The solution for the x-y-z case $\boldsymbol{R}_x(\varphi)\circ\boldsymbol{R}_y(\theta)\circ\boldsymbol{R}_z(\psi)$ is given in Table 6.5, and the solution for the z-y-x case $\boldsymbol{R}_z(\psi)\circ\boldsymbol{R}_y(\theta)\circ\boldsymbol{R}_x(\varphi)$ is given in Table 6.6.

Table 6.5 — Principal factors for the Euler x-y-z convention (Tait-Bryan)

Case	**Principal factors for x-y-z Euler rotation** $\boldsymbol{R}_{z''}(\psi)\circ\boldsymbol{R}_{y'}(\theta)\circ\boldsymbol{R}_x(\varphi)=\boldsymbol{R}_x(\varphi)\circ\boldsymbol{R}_y(\theta)\circ\boldsymbol{R}_z(\psi)$ (all angles modulo 2π, $\boldsymbol{M}=[\ a_{ij}\ ]$)		
$a_{13}\neq\pm1$	$\theta=\arcsin(a_{13})$ [principal value] $-\pi/2<\theta<\pi/2$	$\varphi=\mathrm{arctan2}(-a_{23},a_{33})$	$\psi=\mathrm{arctan2}(-a_{12},a_{11})$
	$\theta=\arcsin(a_{13})$ [π − principal value] $\pi/2<\theta<3\pi/2$	$\varphi=\mathrm{arctan2}(a_{23},-a_{33})$	$\psi=\mathrm{arctan2}(a_{12},-a_{11})$
$a_{13}=+1$	$\theta=\pi/2$	$\varphi=\mathrm{arctan2}(a_{21},-a_{31})-\psi$	any value of ψ
$a_{13}=-1$	$\theta=-\pi/2$	$\varphi=\mathrm{arctan2}(a_{21},a_{31})+\psi$	any value of ψ

In the case $a_{13}\neq\pm1$, arcsin() is multi-valued so that there are two valid solution sets depending on the quadrant selected for arcsine values[21]. The principal value solution is the commonly used one.

In the case $a_{13}=+1$, using the trigonometric identities for the difference of angles and substituting $\sin\theta=1$ and $\cos\theta=0$, the matrix reduces to:

$$\boldsymbol{R}_x(\varphi)\circ\boldsymbol{R}_y\left(\frac{\pi}{2}\right)\circ\boldsymbol{R}_z(\psi)=\begin{pmatrix}0 & 0 & 1\\ \sin(\varphi+\psi) & \cos(\varphi+\psi) & 0\\ -\cos(\varphi+\psi) & \sin(\varphi+\psi) & 0\end{pmatrix}.$$

For this reason only the sum of the other two angles is determined as $\varphi+\psi=\mathrm{arctan2}(a_{21},-a_{31})$. Therefore, all values are valid for ψ if we set $\varphi=\mathrm{arctan2}(a_{21},-a_{31})-\psi$. The case $a_{13}=-1$ is similar to the previous case with the difference of the angles determined by $\varphi-\psi=\mathrm{arctan2}(a_{21},a_{31})$. These two cases correspond to Equation (6.9) and are the gimbal lock cases.

21 Computer library functions such as asin() return the principal value only. The second solution for θ may be obtained by subtracting the principal value from π.

Table 6.6 — Principal factors for the Euler z-y-x convention (Tait-Bryan)

Case	Principal factors for z-y-x Euler rotation $\boldsymbol{R}_{x''}(\varphi) \circ \boldsymbol{R}_{y'}(\theta) \circ \boldsymbol{R}_{z}(\psi) = \boldsymbol{R}_{z}(\psi) \circ \boldsymbol{R}_{y}(\theta) \circ \boldsymbol{R}_{x}(\varphi)$ (all angles modulo 2π, $\boldsymbol{M} =[\ a_{ij}\ ]$)		
$a_{31} \neq \pm 1$	$\theta = \arcsin(-a_{31})$ [principal value] $-\pi/2 < \theta < \pi/2$	$\varphi = \text{arctan2}(a_{32},\ a_{33})$	$\psi = \text{arctan2}(a_{21},\ a_{11})$
	$\theta = \arcsin(-a_{31})$ [π − principal value] $\pi/2 < \theta < 3\pi/2$	$\varphi = \text{arctan2}(-a_{32},\ -a_{33})$	$\psi = \text{arctan2}(-a_{21},\ -a_{11})$
$a_{31} = -1$	$\theta = \pi/2$	$\varphi = \text{arctan2}(a_{12},\ a_{13}) + \psi$	any value of ψ
$a_{31} = +1$	$\theta = -\pi/2$	$\varphi = \text{arctan2}(-a_{12},\ -a_{13}) - \psi$	any value of ψ

In the case $a_{31} = -1$, using the trigonometric identities for the difference of angles and substituting $\sin\theta = 1$ and $\cos\theta = 0$, the matrix reduces to:

$$\boldsymbol{R}_z(\psi) \circ \boldsymbol{R}_y\left(\frac{\pi}{2}\right) \circ \boldsymbol{R}_x(\varphi) = \begin{pmatrix} 0 & \sin(\varphi-\psi) & \cos(\varphi-\psi) \\ 0 & \cos(\varphi-\psi) & -\sin(\varphi-\psi) \\ -1 & 0 & 0 \end{pmatrix}.$$

For this reason only the difference of the other two angles is determined as $\varphi - \psi = \text{arctan2}(a_{12},\ a_{13})$. Therefore, all values are valid for ψ if we set $\varphi = \text{arctan2}(a_{12},\ a_{13}) + \psi$. The case $a_{31} = +1$ is similar to the previous case with the sum of the angles determined by $\varphi + \psi = \text{arctan2}(-a_{12},\ -a_{13})$. These two cases correspond to Equation (6.9) and are the gimbal lock cases.

As seen in the preceding tables, the three angle sequence corresponding to a given rotation or orientation operator is not unique modulo 2π. Two sequences, $(\varphi_1, \theta_1, \psi_1)$ and $(\varphi_2, \theta_2, \psi_2)$ of x-y-z principal factors specify the same operator if they satisfy one the criteria specified in Table 6.6.

Table 6.7 — Equivalence of x-y-z or z-y-x principal factor sequences

Case (equality modulo 2π)	**Criteria for the equivalence of angle sequences $(\varphi_1, \theta_1, \psi_1)$ and $(\varphi_2, \theta_2, \psi_2)$ for principal factor z-y-x or x-y-z sequences**	
$\theta_1 = \theta_2$	$\varphi_1 = \varphi_2,\ \ \psi_1 = \psi_2\ \left[\theta_1 \neq \pm\frac{\pi}{2} \neq \theta_2\right]$	(in)equalities modulo 2π

Case (equality modulo 2π)	**Criteria for the equivalence of angle sequences** $(\varphi_1,\theta_1,\psi_1)$ **and** $(\varphi_2,\theta_2,\psi_2)$ **for principal factor** z-y-x **or** x-y-z **sequences**
$\lvert\theta_1+\theta_2\rvert=\pi$	$\lvert\varphi_2-\varphi_1\rvert=\pi,\ \lvert\psi_2-\psi_1\rvert=\pi\left[\theta_1\neq\pm\frac{\pi}{2}\neq\theta_2\right]$ (in)equalities modulo 2π
$\theta_1=\theta_2=\frac{\pi}{2}$	$\varphi_1+\psi_1=\varphi_2+\psi_2$ x–y–z case $\varphi_1-\psi_1=\varphi_2-\psi_2$ z–y–x case equality modulo 2π
$\theta_1=\theta_2=-\frac{\pi}{2}$	$\varphi_1-\psi_1=\varphi_2-\psi_2$ x–y–z case $\varphi_1+\psi_1=\varphi_2+\psi_2$ z–y–x case equality modulo 2π

6.8.2 Matrix and axis-angle

Given a rotation matrix $\boldsymbol{R}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$, the corresponding axis-angle representation $(\boldsymbol{n},\theta)$ is determined using the procedure in 6.7.2.

An axis-angle rotation $(\boldsymbol{n},\theta)$, with $\boldsymbol{n}=\begin{pmatrix}n_1 & n_2 & n_3\end{pmatrix}^{\mathrm{T}}$, is converted to rotation matrix $\boldsymbol{R}$, using the matrix form of Rodrigues' rotation formula (Equation (6.3)).

$$\begin{aligned}\boldsymbol{R}&=\left[\boldsymbol{I}_{3\times3}+\sin(\theta)\boldsymbol{S}_{\boldsymbol{n}}+\left(1-\cos(\theta)\right)\boldsymbol{S}_{\boldsymbol{n}}^2\right]\\&=\left[\cos(\theta)\boldsymbol{I}_{3\times3}+\left(1-\cos(\theta)\right)\boldsymbol{n}\otimes\boldsymbol{n}+\sin(\theta)\boldsymbol{S}_{\boldsymbol{n}}\right]\end{aligned}\tag{6.16}$$

where:

$\boldsymbol{S}_{\boldsymbol{n}}=\begin{pmatrix}0&-n_3&n_2\\n_3&0&-n_1\\-n_2&n_1&0\end{pmatrix}$ is the skew-symmetric matrix associated with $\boldsymbol{n}=\begin{pmatrix}n_1&n_2&n_3\end{pmatrix}^{\mathrm{T}}$ and

$\boldsymbol{n}\otimes\boldsymbol{n}=\begin{pmatrix}n_1n_1&n_1n_2&n_1n_3\\n_2n_1&n_2n_2&n_2n_3\\n_3n_1&n_3n_2&n_3n_3\end{pmatrix}$ is the outer-product of $\boldsymbol{n}$ with $\boldsymbol{n}$.

The equation expands to yield matrix elements:

$$\boldsymbol{R}=\begin{pmatrix}(1-\cos\theta)n_1^2+\cos\theta & (1-\cos\theta)n_1n_2-n_3\sin\theta & (1-\cos\theta)n_1n_3+n_2\sin\theta\\(1-\cos\theta)n_2n_1+n_3\sin\theta & (1-\cos\theta)n_2^2+\cos\theta & (1-\cos\theta)n_2n_3-n_1\sin\theta\\(1-\cos\theta)n_3n_1-n_2\sin\theta & (1-\cos\theta)n_3n_2+n_1\sin\theta & (1-\cos\theta)n_3^2+\cos\theta\end{pmatrix}\tag{6.17}$$

6.8.3 Axis-angle and quaternion

A rotation in axis-angle form $(\boldsymbol{n},\theta)$ corresponds to unit quaternion $\boldsymbol{q} = \left(\cos(\theta/2), \sin(\theta/2)\boldsymbol{n}\right)$.

A unit quaternion corresponds to axis-angle form $(\boldsymbol{n},\theta)$ computed as in Equation (6.11).

6.8.4 Matrix and quaternion

The matrix $\boldsymbol{M}$ corresponding to a unit quaternion $\boldsymbol{q} = (e_0, \boldsymbol{e})$, $\boldsymbol{e} = (e_1, e_2, e_3)^{\mathrm{T}}$ is

$$\boldsymbol{M} = \begin{pmatrix} 1-2\left(e_2^2+e_3^2\right) & 2\left(e_1e_2-e_0e_3\right) & 2\left(e_1e_3+e_0e_2\right) \\ 2\left(e_1e_2+e_0e_3\right) & 1-2\left(e_1^2+e_3^2\right) & 2\left(e_2e_3-e_0e_1\right) \\ 2\left(e_1e_3-e_0e_2\right) & 2\left(e_2e_3+e_0e_1\right) & 1-2\left(e_1^2+e_2^2\right) \end{pmatrix} \tag{6.18}$$

The quaternion $\boldsymbol{q}$ corresponding to a rotation matrix $\boldsymbol{M} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is computed as follows:

$$e_0^2 = \tfrac{1}{4}\left(1+\mathrm{Trace}(\boldsymbol{R})\right) = \tfrac{1}{4}\left(1+a_{11}+a_{22}+a_{33}\right)$$

if $e_0^2 > 0$,

$$\boldsymbol{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \frac{1}{4e_0}\begin{pmatrix} a_{32}-a_{23} \\ a_{13}-a_{31} \\ a_{21}-a_{12} \end{pmatrix},$$

else $e_0 = 0$,

$e_1^2 = -\tfrac{1}{2}\left(a_{22}+a_{33}\right)$,

if $e_1^2 > 0$, $e_2 = \dfrac{a_{12}}{2e_1}$, $e_3 = \dfrac{a_{13}}{2e_1}$,

else $e_1 = 0$,

$e_2^2 = \tfrac{1}{2}\left(1-a_{33}\right)$,

if $e_2^2 > 0$, $e_3 = \dfrac{a_{23}}{2e_2}$

else $e_2 = 0$, $\quad e_3 = 1$.

A rotationally equivalent quaternion is $-\boldsymbol{q}$.

6.8.5 Euler angle conventions and quaternions

The principal rotations (see 6.7.4.1) correspond to the following quaternions:

$$\boldsymbol{R}_x(\alpha) \leftrightarrow \left(\cos(\alpha/2), \sin(\alpha/2)\boldsymbol{x}\right)$$
$$\boldsymbol{R}_y(\beta) \leftrightarrow \left(\cos(\beta/2), \sin(\beta/2)\boldsymbol{y}\right)$$
$$\boldsymbol{R}_z(\gamma) \leftrightarrow \left(\cos(\gamma/2), \sin(\gamma/2)\boldsymbol{z}\right)$$

For each Euler angle convention, multiply the corresponding quaternions in body-fixed composition order. Terms in the resulting product may be simplified using the orthonormal property of the vector set $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$, and various trigonometric identities.

For the Euler angle z-x-z convention, the quaternion $\boldsymbol{q}$ corresponding to $\boldsymbol{R}_{z''}(\gamma) \circ \boldsymbol{R}_{x'}(\beta) \circ \boldsymbol{R}_{z}(\alpha) = \boldsymbol{R}_{z}(\alpha) \circ \boldsymbol{R}_{x}(\beta) \circ \boldsymbol{R}_{z}(\gamma)$ is:

$$\boldsymbol{q} = \left(\cos\left(\frac{\alpha}{2}\right), \sin\left(\frac{\alpha}{2}\right)\boldsymbol{z}\right)\left(\cos\left(\frac{\beta}{2}\right), \sin\left(\frac{\beta}{2}\right)\boldsymbol{x}\right)\left(\cos\left(\frac{\gamma}{2}\right), \sin\left(\frac{\gamma}{2}\right)\boldsymbol{z}\right).$$

Multiplied out, the expression reduces to:

$$\boldsymbol{q} = (e_0, \boldsymbol{e})$$

where:

$$e_0 = \cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right),$$

$$\boldsymbol{e} = \left(\cos\left(\frac{\alpha-\gamma}{2}\right)\sin\left(\frac{\beta}{2}\right),\ \sin\left(\frac{\alpha-\gamma}{2}\right)\sin\left(\frac{\beta}{2}\right),\ \sin\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\right)$$

For the Euler angle x-y-z convention (Tait-Bryan angles), the quaternion $\boldsymbol{q}$ corresponding to $\boldsymbol{R}_{z''}(\psi) \circ \boldsymbol{R}_{y'}(\theta) \circ \boldsymbol{R}_{x}(\varphi) = \boldsymbol{R}_{x}(\varphi) \circ \boldsymbol{R}_{y}(\theta) \circ \boldsymbol{R}_{z}(\psi)$ is:

$$\boldsymbol{q} = \left(\cos\left(\frac{\varphi}{2}\right), \sin\left(\frac{\varphi}{2}\right)\boldsymbol{x}\right)\left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\boldsymbol{y}\right)\left(\cos\left(\frac{\psi}{2}\right), \sin\left(\frac{\psi}{2}\right)\boldsymbol{z}\right).$$

Multiplied out, the expression reduces to:

$$\boldsymbol{q} = (e_0, \boldsymbol{e}) = (e_0, e_1, e_2, e_3)$$

where:

$$e_0 = \cos(\varphi/2)\cos(\theta/2)\cos(\psi/2) - \sin(\varphi/2)\sin(\theta/2)\sin(\psi/2)$$

$$e_1 = \cos(\varphi/2)\sin(\theta/2)\sin(\psi/2) + \sin(\varphi/2)\cos(\theta/2)\cos(\psi/2)$$

$$e_2 = \cos(\varphi/2)\sin(\theta/2)\cos(\psi/2) - \sin(\varphi/2)\cos(\theta/2)\sin(\psi/2)$$

$$e_3 = \cos(\varphi/2)\cos(\theta/2)\sin(\psi/2) + \sin(\varphi/2)\sin(\theta/2)\cos(\psi/2)$$

For the Euler angle z-y-x convention (Tait-Bryan angles), the quaternion $\boldsymbol{q}$ corresponding to $\boldsymbol{R}_{x''}(\varphi) \circ \boldsymbol{R}_{y'}(\theta) \circ \boldsymbol{R}_{z}(\psi) = \boldsymbol{R}_{z}(\psi) \circ \boldsymbol{R}_{y}(\theta) \circ \boldsymbol{R}_{x}(\varphi)$ is:

$$\boldsymbol{q} = \left(\cos\left(\frac{\psi}{2}\right), \sin\left(\frac{\psi}{2}\right)\boldsymbol{z}\right)\left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\boldsymbol{y}\right)\left(\cos\left(\frac{\varphi}{2}\right), \sin\left(\frac{\varphi}{2}\right)\boldsymbol{x}\right).$$

Multiplied out, the expression reduces to:

$$\boldsymbol{q} = (e_0, \boldsymbol{e}) = (e_0, e_1, e_2, e_3)$$

where:

$$e_0 = \cos(\psi/2)\cos(\theta/2)\cos(\varphi/2) + \sin(\psi/2)\sin(\theta/2)\sin(\varphi/2)$$

$$e_1 = \cos(\psi/2)\cos(\theta/2)\sin(\varphi/2) - \sin(\psi/2)\sin(\theta/2)\cos(\varphi/2)$$

$$e_2 = \cos(\psi/2)\sin(\theta/2)\cos(\varphi/2) + \sin(\psi/2)\cos(\theta/2)\sin(\varphi/2)$$

$$e_3 = \sin(\psi/2)\cos(\theta/2)\cos(\varphi/2) - \cos(\psi/2)\sin(\theta/2)\sin(\varphi/2)$$

To convert a unit quaternion $\boldsymbol{q} = (e_0, \boldsymbol{e}) = (e_0, e_1, e_2, e_3)$ to the Euler angle z–x–z convention $\boldsymbol{R}_z(\alpha) \circ \boldsymbol{R}_x(\beta) \circ \boldsymbol{R}_z(\gamma)$, compute as follows:

if $0 < (e_1^2 + e_2^2) < 1$:

$$\alpha = \text{arctan2}\left((e_1e_3 + e_0e_2), -(e_2e_3 - e_0e_1)\right)$$
$$\beta = \arccos\left(1 - 2(e_1^2 + e_2^2)\right) \qquad \text{principal value: } 0 < \beta < \pi$$
$$\gamma = \text{arctan2}\left((e_1e_3 - e_0e_2), (e_2e_3 + e_0e_1)\right)$$

if $(e_1^2 + e_2^2) = 0$: $\beta = 0$ and $\alpha + \gamma = \text{arctan2}\left((e_1e_2 - e_0e_3), \tfrac{1}{2} - (e_2^2 + e_3^2)\right)$.

if $(e_1^2 + e_2^2) = 1$: $\beta = \pi$ and $\alpha - \gamma = \text{arctan2}\left((e_1e_2 - e_0e_3), \tfrac{1}{2} - (e_2^2 + e_3^2)\right)$.

The solution in the first case is not unique, see Table 6.4. The last two cases are Euler angle gimbal lock cases.

To convert a unit quaternion $\boldsymbol{q} = (e_0, \boldsymbol{e}) = (e_0, e_1, e_2, e_3)$ to the Euler angle x-y-z convention (Tait-Bryan angles) $\boldsymbol{R}_x(\varphi) \circ \boldsymbol{R}_y(\theta) \circ \boldsymbol{R}_z(\psi)$, compute as follows.

If $2(e_1e_3 + e_0e_2) \neq \pm 1$:

$$\varphi = \text{arctan2}\left((e_2e_3 - e_0e_1),\ \tfrac{1}{2} - (e_1^2 + e_2^2)\right)$$
$$\theta = \arcsin\left(2(e_1e_3 + e_0e_2)\right) \qquad \text{principal value: } -\pi/2 < \theta < \pi/2$$
$$\psi = \text{arctan2}\left(-(e_1e_2 - e_0e_3),\ \tfrac{1}{2} - (e_2^2 + e_3^2)\right)$$

If $2(e_1e_3 + e_0e_2) = +1$: $\theta = -\pi/2$ and $\varphi + \psi = \text{arctan2}\left((e_1e_2 + e_0e_3), -(e_1e_3 - e_0e_2)\right)$.

If $2(e_1e_3 + e_0e_2) = -1$: $\theta = \pi/2$ and $\varphi - \psi = \text{arctan2}\left((e_1e_2 + e_0e_3), (e_1e_3 - e_0e_2)\right)$.

The solution in the first case is not unique, see Table 6.7. The last two cases are Euler angle gimbal lock cases.

To convert a unit quaternion $\boldsymbol{q} = (e_0, \boldsymbol{e}) = (e_0, e_1, e_2, e_3)$ to the Euler angle z-y-x convention (Tait-Bryan angles) $\boldsymbol{R}_z(\psi) \circ \boldsymbol{R}_y(\theta) \circ \boldsymbol{R}_x(\phi)$, compute as follows.

If $2(e_1e_3 - e_0e_2) \neq \pm 1$:

$$\varphi = \text{arctan2}\left((e_2e_3 + e_0e_1),\ \tfrac{1}{2} - (e_1^2 + e_2^2)\right)$$
$$\theta = \arcsin\left(-2(e_1e_3 - e_0e_2)\right) \qquad \text{principal value: } -\pi/2 < \theta < \pi/2$$
$$\psi = \text{arctan2}\left((e_1e_2 + e_0e_3),\ \tfrac{1}{2} - (e_2^2 + e_3^2)\right)$$

If $2(e_1e_3 - e_0e_2) = +1$: $\theta = -\pi/2$ and $\varphi + \psi = \text{arctan2}\left((e_1e_2 - e_0e_3), (e_1e_3 + e_0e_2)\right)$.

If $2(e_1e_3 - e_0e_2) = -1$: $\theta = \pi/2$ and $\varphi - \psi = \text{arctan2}\left((e_1e_2 - e_0e_3), (e_1e_3 + e_0e_2)\right)$.

The solution in the first case is not unique, see Table 6.7. The last two cases are Euler angle gimbal lock cases.

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